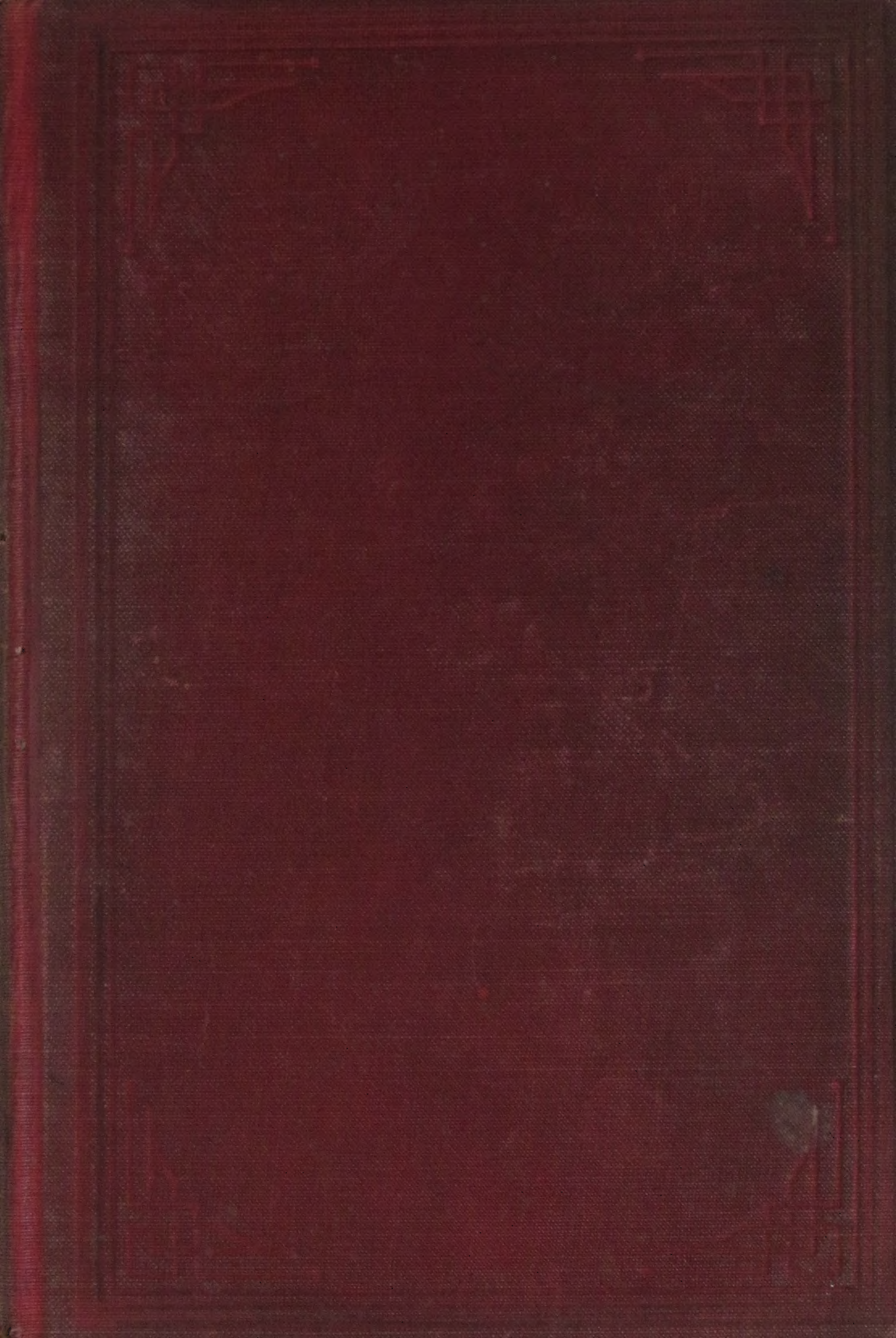

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ALGEBRA

BY

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PREFACE.

This book is designed especially for students preparing to enter college. It endeavors to cover with thoroughness the exposition and application of the elementary principles of algebra, but omits entirely all those topics ordinarily included in the advanced algebra of the college course. It is believed that its method of treatment is such as to make it available for use in High Schools aiming at a thorough mastery of the elementary portion of the subject.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. **A quantity** is anything that can be increased or diminished. Finite quantities can be **measured.**

2. To **measure** a quantity is to find how many times it contains another quantity of the same kind. This latter quantity is called the **unit of measure.** Its amount is arbitrary. An inch, a foot, a yard are units of length; an ounce, a pound, a ton are units of weight.

3. **Number** shows how many times a quantity contains its unit of measure. It is quantity determined and expressed.

4. Numbers are **concrete** and **abstract**; concrete, when attached to a particular kind of quantity, as 6 days; abstract, when not thus attached, as 4, 6.

5. In **Arithmetic** numbers are represented by figures.

6. In **Algebra** numbers are represented by figures and letters.

7. **Algebra** differs from arithmetic:

1. In the use of symbolic language which enables general as well as specific quantities to be considered, thus generalizing arithmetic.

2. In that the relation of quantities rather than the computation of their numerical value is made the chief object of inquiry.

8. Algebra may then be defined as that branch of mathematics in which quantities are represented by symbols and their relations investigated.

9. Algebraic Symbols as a whole may be classified as follows:

1. Symbols denoting quantity.
2. Symbols indicating relation.
3. Symbols indicating operation.
4. Symbols indicating abbreviation.

QUANTITIES AND QUANTITATIVE SYMBOLS.

10. Known quantities are those whose values are given. They are represented by figures or the first letters of the alphabet. Accented letters are often used where quantities are in the same class, as a', read a prime; a'', read a second; a''', read a third, and so on. Subscript numbers are sometimes employed, as a_1, read a sub one; a_2, read a sub two, and so on. Initial letters may also be used to advantage, as r for radius, c for circumference, etc.

Concerning the symbolic letters which represent known quantities it is important to note:

1. The value they represent is *any* value.

2. They are conceived as retaining the *same* value throughout the process of deriving an answer or formula.

3. A particular value may be given to any letter at any time during the progress of the work. This,

though seldom done, is in rare cases of great convenience.

4. Particular values may be assigned the letters at the end of the work. The numerical value of the answer will be thus determined.

5. It is not necessary to assign to the letters particular values either during or at the end of the work.

11. Unknown quantities are those whose values though fixed are to be ascertained. They are generally, but not always, represented by the last letters of the alphabet, x, y, z, etc.

12. Quantities greater than any assignable quantity are represented by the symbol ∞, called infinity.

13. Absence of quantity or a quantity less than any assignable quantity is represented by the symbol 0, called zero.

14. Positive and negative quantities indicate direction from zero. They stand in opposition to each other and represent opposite states or qualities. They lie on opposite sides of the zero line.

Thus,

ZERO LINE.

If negative quantities represent	Then positive quantities represent
left	right
above	below
forward	backward
losses	gains
liabilities	assets
past	future

Positive and negative quantities are distinguished from each other by the signs + and —. Quantitative symbols with unassigned values may belong to either class. Each class contains all the numerical quantities between zero and infinity, the only difference being in sign. As arithmetic recognizes only absolute value or amount of quantity independent of the sign, each class is considered as arithmetically increasing from zero to infinity. In algebra, however, quantities are considered to increase as we pass from negative infinity to zero and thence to positive infinity. Proceeding in the opposite direction, quantities are viewed as decreasing. Counting backward along the series of positive numbers, we are evidently subtracting or decreasing quantity. Counting from 3 to 0, we subtract 3 and thus 0 is less than 3. Passing through 0 and continuing to count in the same direction, we are evidently still subtracting. — 3 is thus clearly *less* than 0. Algebraically considered then quantities decrease as we pass from positive infinity to negative infinity and increase as we go in the opposite direction.

We thus see that

Passing from $-\infty$ to 0 means algebraic increase, arithmetical decrease.

Passing from 0 to $+\infty$ means algebraic and arithmetical increase.

Passing from $+\infty$ to 0 means algebraic and arithmetical decrease.

Passing from 0 to $-\infty$ means algebraic decrease, arithmetical increase.

SYMBOLS INDICATING RELATION.

The symbols indicating relation are as follows:

15. Sign of equality $=$. This is read *equals* or *is equal to*. It signifies that the quantities between which it is placed are equal. The expression as a whole is called an equation. The two parts into which it is separated by the sign are called members of the equation.

Example,
Equation $c = d$ (read c equals d or c is equal to d).
First member c
Second member d
Equality sign $=$

16. Signs of inequality $>$ **and** $<$. These are read *greater than* and *less than* respectively. Thus,

$6 > 5$, read six is greater than five.
$a > b$, read a is greater than b.
$5 < 6$, read five is less than six.
$b < a$, read b is less than a.

The opening of the angle is always toward the greater quantity. The expression as a whole is called an inequality; its parts are called members of the inequality.

17. The Ratio Sign $:$. This is read *is to* or *to*. It signifies the ratio of the two quantities between which it is placed. Thus,

$3 : 4$, read 3 is to 4.
$a : b$, read a is to b.

These ratios may be written $\frac{3}{4}$ and $\frac{a}{b}$.

18. The sign of proportion :: or =. This is read *as* or *equals*. It signifies that the ratios between which it is placed are equal, thus constituting a proportion.

$$\left.\begin{array}{c} a : b :: c : d \\ \text{or} \\ a : b = c : d \end{array}\right\} \text{read} \left\{\begin{array}{c} a \text{ is to } b \textit{ as } c \text{ is to } d \\ \text{or} \\ a \text{ to } b \textit{ equals } c \text{ to } d. \end{array}\right.$$

Expressing the ratios in their fractional form the proportion may be written

$$\frac{a}{b} = \frac{c}{d}.$$

SYMBOLS INDICATING OPERATIONS.

The following are the symbols of operation:

19. The Sign of Addition, +, read plus.

20. The Sign of Subtraction, —, read minus.

21. The Double Sign, ± and ∓, read plus or minus and minus or plus respectively.

22. The Sign of Multiplication, ×, read multiplied by.

The quantities which multiplied together form a product are called factors of that product; if expressed in numbers they are called *numerical* factors; if expressed in letters, *literal* factors. When literal factors or a figure and literal factors are multiplied, the sign of multiplication is omitted.

Thus $7 \times a \times b \times c$ is written $7abc$.

Care should be taken never to omit the sign of multiplication when such omission will lead to ambiguity or a change of meaning. 3×2 would, by omitting the sign, become 32, a radical change in significance.

23. The Sign of Division, ÷ , read divided by.

Division is often indicated by placing the quantity to be divided over the divisor with a horizontal line between them.

Thus, $\frac{a}{b}$ signifies the same as $a \div b$.

24. The Exponential Sign is the symbol written to the right and a little above a quantity to show how many times that quantity is taken as a factor. The resulting product is called the power of the quantity. The symbol is called an exponent. a^n is read a to the nth power, indicating that a is taken n times as a factor, a and n having any values. The reading is sometimes abbreviated, thus a^2 may be read a to the second power or a square; a^3, a to the third power or a cube; a^4, a to the fourth power or a fourth power and so on.

If no exponent is expressed *one* is understood. Thus a is the same as a^1.

25. The Radical Sign, $\sqrt{}$, placed over a quantity donotes that a root of that quantity is desired. The particular root sought is indicated by a figure or letter written to the left and above the sign, which letter or figure is called the **Index of the Root.** Thus,

$\sqrt[3]{3b}$ indicates the third or cube root of $3b$.

$\sqrt[4]{5c}$ indicates the fourth root of $5c$.

$\sqrt[n]{7d}$ indicates the nth root of $7d$.

The index in the case of the square root is generally omitted, thus,

$\sqrt{2}$ indicates the second or square root of 2.

26. The Root of a quantity is that quantity which, taken as a factor the number of times indicated by the index, will produce the given quantity.

Thus, 2 taken as a factor three times produces 8, hence $\sqrt[3]{8} = 2$.

SYMBOLS INDICATING ABBREVIATION.

The following symbols indicate abbreviation:

27. The Signs of Aggregation, the parenthesis $(\quad)$, vinculum $\overline{\qquad}$, bracket $[\quad]$, brace $\{\quad\}$, and bar $|$, signify that the quantities thus connected are to be treated as a single quantity.

Thus $(a + b + c)\,d$ signifies that the three quantities a, b, and c united into *one quantity* are to be multiplied by d. The union of quantities into a single quantity to be treated *as such* is of vital importance. The use of the signs indicative of such aggregation must be thoroughly mastered in order to reach a comprehensive grasp of algebraic generalization.

Two or more of these signs are often used in the same expression. Thus $\{\, a + b\,[\, c + (d + e)\,]f \,\}\, x$.

28. The Sign of Deduction $\therefore$ signifies *hence* or *therefore*.

29. The Sign of Continuation . . . . . . is read *and so on* or *and so on to*.

Thus we can read 1, 2, 3, 4, 5 . . . . . .
and 1, 2, 3, 4, 5, ... 10.

ALGEBRAIC EXPRESSIONS.

30. An Algebraic Expression or simply an **Expression** is a collection of quantitative algebraic symbols. Thus a, $3x$, $x+y$, $x-3y$ $(a+b)+c$ are algebraic expressions. Values may be assigned to the letters used and the indicated operations performed. The result obtained will be the numerical value of the expression.

31. The Terms of an algebraic expression are its parts, which, added or subtracted, form the expression. Thus,

$2a$, $5b$, and $7c$ are terms of the expression $2a-5b+7c$.

$(a-b)$ and $-c$ are terms of the expression $(a-b)-(-c)$.

$\sqrt{x+y}$ and $+n$ are terms of the expression $\sqrt{x+y}+n$.

32. A Simple Term is one which, except in index or exponent, contains no connective + or — sign.

33. A Complex Term is one which is treated as a single quantity, and contains two or more simple terms connected by the sign + or —. Thus,

$(a+b)$, $(a+b)c$, $(a+b)(c+d)$, $\sqrt{a+b+c}$, $(a+b)^{x+y}$ are complex terms.

34. A Positive Term is one preceded by the + sign (of operation), expressed or understood; as $3a$, $+5b$, $+(a+b)$ and $+(-3x)$.

35. A Negative Term is one preceded by the minus sign (of operation); as $-5a$, $-2a\sqrt{3}$, $-\frac{3ab}{c}$, and $-c(x+y)$.

36. **A Monomial** consists of one term either simple or complex.

37. **A Polynomial** consists of two or more terms, any one of which may be either simple or complex.

38. **A Binomial** is a polynomial of two terms; as $x + \frac{a+b}{c}$.

39. **A Trinomial** is a polynomial of three terms; as $x + c(a + b) - \sqrt{x - y}$.

40. When the literal factors of a simple term are single letters with positive whole numbers for exponents, the **degree of the term** is equal to the sum of these exponents. Thus,

$3b$ is of the first degree.
$-4ab$ is of the second degree.
$5a^2c$ is of the third degree.
$5xy^2z^3$ is of the sixth degree.

41. **The degree of an expression** composed of such simple terms is that of the term or terms of highest degree.

42. **Homogeneous Terms** are those of the same degree.

43. **A Product** is the result obtained by multiplying together two or more quantities. These quantities, as already stated, are called *factors*. It is evident that any one of them will exactly divide the product. A factor may be a monomial or a polynomial; if a polynomial, one of the signs of aggregation should be used. Thus $(a+b+c)\,d$, $(\sqrt{a+b}+d)\,x$, $\left(\frac{a+b}{c}+d\right)ex$.

44. A Prime Factor is one that cannot be resolved into other factors. It is therefore divisible only by itself and unity.

45. A Composite Factor is the product of two or more factors, neither of which is unity. Thus $5a$ is a composite factor of $5ab$.

46. Any factor, either prime or composite, is the **coefficient** of the product of the remaining factors of a quantity. Thus in $7ab$, 7 is the coefficient of ab; $7a$ of b; $7b$ of a and a of $7b$. In $(a-b)(c-d)(x-y+u)$ any one of the three factors is the coefficient of the product of the other two and the product of any two is the coefficient of the third.

If the coefficient is a number, it is called a *numerical coefficient*; if it is composed of letters, or letters and figures, it is called a *literal coefficient*.

47. A Common Factor of two or more quantities is a quantity which is a factor of each of them.

48. The Highest Common Factor of two or more quantities is their common factor of highest degree, with the greatest numerical coefficient. Such highest common factor is always the product of the prime factors common to the quantities.

Thus a^2b is the highest common factor of $7a^3b$ and $8a^2b^3$; $8a^4b^5$ is the highest common factor of $16a^4b^7$ and $24a^8b^5$, being higher than the common factor $4a^4b^5$ in that its numerical coefficient is greater.

49. An Axiom is a self-evident truth.

50. A Theorem is a truth to be proven.

51. A Problem is a question proposed for solution.

52. A Proposition is a general term for a theorem or a problem.

53. Examples are principles illustrated and applied.

The essentials of success in algebraic work are:

A clear understanding of definitions.

A thorough mastery of principles.

Care, neatness and skill in algebraic operations.

An infallible symptom of failure is the substitution of mere mechanical processes for genuine mental mastery.

54. Translate into common language the following algebraic expressions and find the numerical value of each when $a = 4$, $b = 5$, $m = 3$, $n = 10$, $x = 2$, $z = 7$:

1. $\frac{a+b}{m} - \frac{n}{b} + z^2.$

Ans. The quotient arising from dividing the sum of a and b by m, diminished by the quotient of n divided by b, and increased by the square of z, equals, for the given values, $\frac{4+5}{3} - \frac{10}{5} + 7^2 = 3 - 2 + 49 = 50.$

2. $n^2 - m^2 + ab - xz.$
3. $x^2(z - m) - bm.$
4. $(n - m)(a + 2b).$
5. $\frac{3a + n - x}{2m - x}.$
6. $(z - x)a + (m + n - b)^2.$

7. $\sqrt{n + z + 2a}$.

8. $\sqrt[3]{3m + ab - x} = \sqrt{xz - b}$.

9. $[a + b - (m - x)]\sqrt{5b - 3m} = az + 4$.

55. Determine the numerical value of the following when $x = 6$, $y = 5$, $z = 4$, $a = 3$, $b = 2$, $c = 1$:

1. $(z - a)\sqrt{3x - \sqrt{2a - b}}$.

2. $\sqrt{\frac{xz + c}{z}} + y + (2x - 3a)^2$.

3. $\frac{1}{y}\left\{bxy - [a + z - (y - \overline{b - c})] - b\right\}$.

4. $\frac{1}{b^2}\left\{\sqrt{z} + [\sqrt{3a} - (a - b)^2]\right\} + ab^2c^3$.

56. Express in algebraic language the following:

1. The sum of x increased by three times z.

2. The product of the square of b multiplied by the sum of x and y divided by the square root of the sum of 3 times m and 4 times a.

3. The square root of the product of the sum of a and b by the difference of m and n.

4. The cube root of 9 times the sum of m and n diminished by the square of three times the sum of z and k.

5. The quotient obtained by dividing the sum of a and b by the cube of a quantity composed of the sum of the cube of x and the square of y diminished by the square root of m.

6. The cube root of the quantity, the square root of 5 times x diminished by the fourth root of 7 times the

cube of y and by 6 times the sum of a and b is equal to the quantity, m increased by n multiplied by the quantity x diminished by y.

7. The quantity composed of the sum of m and n diminished by the difference of the square of p and the cube of q multiplied by the square root of the quotient obtained by dividing the difference of x and y by the sum of m and n is equal to the square root of the quantity x diminished by the square root of 3 times a.

CHAPTER II.

ADDITION AND SUBTRACTION.

ADDITION.

57. Algebraic Addition is the process of uniting two or more quantities into one equivalent expression of the simplest form. The result is called the algebraic sum.

Addition in algebra has a wider significance than in arithmetic. It not only includes the addition of positive quantities and negative quantities separately, but the finding of the aggregate value of quantities, a part of which are positive and the remainder negative.

The sum of 3 positive units, 4 positive units, and 5 positive units is 12 positive units, therefore $3 + 4 + 5 = 12$.

The sum of 3 negative units, 4 negative units, and 5 negative units is 12 negative units, therefore $(-3) + (-4) + (-5) = -12$.

A force of 8 pounds acts upward; a force of 3 pounds acts downward. The former is designated as positive, the latter consequently as a negative force. The combined (added) effect is a positive force of 5 pounds, therefore $+8$ and -3 added $= +5$.

A man owns three pieces of property worth $1000, $500, and $250 respectively. He owes in all to different parties $300, $80, and $50. He would be accounted worth $1320. That is, designating his assets as positive and his liabilities as negative quantities, $1000 + 500 + 250 + (-300) + (-80) + (-50) = 1320$.

A thermometer indicates 0°. It rises 12°, then falls 15°, then rises 3° and again falls 6°. The aggregate result is a fall of 6°. Therefore $12 + (-15) + 3 + (-6) = -6$.

It is evident from the foregoing:

First, that the algebraic sum of quantities with like signs equals their arithmetical sum with the same sign.

Second, that the algebraic sum of quantities, part of which are positive and the remainder negative, is the arithmetical difference between *the sum of the positive* and *the sum of the negative quantities* with the sign of the greater sum.

58. The following laws proven in arithmetic, hold when numbers are represented by algebraic symbols:

1. The sum of any number of quantities will be the same in whatever order the quantities are taken.

The sum of 3, 5, 7, and 11 is 26. A change in the order of adding will not change the final sum. Using letters to represent numbers, the value of $a + b$ is evidently the same as $b + a$ and $a - b + c$ the same as $a + c - b$.

2. Adding to any quantity the sum of any number of quantities is the same as adding those quantities separately.

Adding 24 to a given quantity is the same as adding to that quantity $2 + 4 + 6 + 12$ or $6 + 18$. Adding to a quantity $(b + c + d)$ is the same as adding b, c, and d separately and in any order.

59. Addition of Monomials. In the addition of algebraic monomials, there are two cases:

1. When the quantities have no common factor.

2. When the quantities have a prime or composite common factor.

CASE I.

When the quantities to be added have no common factor.

RULE.

Connect the quantities by their signs.

EXAMPLES.

1. Find the sum of $3b, -5x$, and $4y$.

Ans. $3b - 5x + 4y$.

2. Find the sum of $2a, -5bc, 4dx$, and $-6yz$.

Ans. $2a - 5bc + 4dx - 6yz$.

CASE II.

When the quantities to be added have a common factor.

The sum can be obtained in this as in the preceding case by simply connecting the quantities by their signs. The presence of a common factor permits, however, such a combination of terms as puts the result in much simpler form.

If we regard this common factor as a unit of measure, its coefficients will represent the number of times it is contained in the successive quantities. The algebraic sum of these coefficients will express the number of times this measuring unit is contained in the quantities taken together. This sum, or new coefficient, multiplied by the measuring unit is the sum of the quantities.

Thus, in adding the two quantities, 7 dollars and 3 dollars, one dollar is the unit of measure. It is contained 7 times in the first quantity, 3 times in the second quantity, and 7 + 3 or 10 times in their sum. Ten times one dollar or ten dollars is therefore the sum of the quantities.

Adding the three quantities, $9x$, $5x$, and $-3x$, x is the unit of measure. It is contained respectively 9, 5, and -3 times in the quantities and $9 + 5 - 3$ times or 11 times in the quantities taken together. Eleven times x or $11x$ is therefore the sum of the quantities.

The *common factor*, taken as the measuring unit, is generally but not always the *highest* common factor. The factor chosen is appropriately called the **Unit of Addition.**

We have then for the addition of quantities containing a common factor the following

RULE.

Write as the coefficient of the chosen unit of addition the algebraic sum of its coefficients in the quantities to be added.

In solving examples attention is called:

First. To the importance of carefully inspecting the quantities to be added before deciding upon the unit of addition.

Second. To the necessity of inclosing all polynomial coefficients in a parenthesis.

EXAMPLES.

1. Determine the unit of addition in the following:

2 books, 3 books, and 7 books.

$5a$, $6a$, and $7a$.

$5bc$, $2b^2c$, mbc.

$5\sqrt{x}$, $3\sqrt{x}$, and $7a\sqrt{x}$.

$(a+x)^2$, $5(a+x)^2$, and $6b(a+x)^2$.

2. Find the sum of $3xy$, $4xy$, and $6xy$.

3. Find the sum of $-6a^2b$, $-2a^2b$, $-a^2b$, and $-8a^2b$. Ans. $-17a^2b$.

4. Find the sum of $3a^{-2}$, $5a^{-2}$, and $4a^{-2}$.
Ans. $12a^{-2}$.

5. Find the sum of ax, bx, and cx.
Ans. $(a+b+c)x$.

6. Find the sum of $6max$, $3nax$, $12pax$, and $9vax$.

7. Find the sum of $b(m-\sqrt{m})$, $2a(m-\sqrt{m})$, $(m-\sqrt{m})$, and $c(m-\sqrt{m})$.

8. Find the sum of a^2cx, $-4a^2cx$, $6a^2cx$, and $-a^2cx$. Ans. $2a^2cx$.

9. Find the sum of $a(x^2-y^2)$, $b(x^2-y^2)$ and $-c(x^2-y^2)$.

10. Find the sum of $\frac{1}{2}\sqrt[3]{a-3b}$, $-\frac{1}{4}\sqrt[3]{a-3b}$, $\frac{1}{8}\sqrt[3]{a-3b}$, and $\frac{1}{16}\sqrt[3]{a-3b}$.

11. Find the sum of $3a(x-y)^{-3}$, $2b(x-y)^{-3}$, and $(x-y)^{-3}$.

It often happens that only a part of the monomials to be added have a common factor. It is then necessary to use both of the foregoing rules. Thus:

$$3m + 4m + 5m + 7b = 12m + 7b.$$

EXAMPLES.

1. Find the sum of $3ab$, $4mn$, $2ab$, and $3mn$.

2. Find the sum of $2a^{-2}$, $5b^{-3}$, $6a^{-2}$, and $-b^{-3}$.

3. Find the sum of $3amn$, $-2bmn$, $7pxy$, $5cmn$, and $9qxy$.

4. Find the sum of $3(x+y)$, $5(x-y)$, $-3(x-y)$, and $-(x+y)$.

5. Find the sum of $5a\sqrt{m-n}$, $6b\sqrt{z+a}$, $-2c\sqrt{m-n}$, and $\sqrt{z+a}$.

60. Addition of Polynomials

RULE.

Find the sum of the terms having common factors, continuing the process until all such terms have been united. To the result annex the remaining terms by their proper signs.

Thus,

Add $ax + 5y + c$, $6ax - y$, and $2bx + 2y - d$.

$$\begin{array}{c}
ax + 5y + c \\
6ax - y \\
2bx + 2y - d \\
\hline
(7a + 2b)\,x + 6y + c - d
\end{array}$$

Add $mn - pq + xy$, $xn + mq - py$ and $xq - pn + my$.

$$\begin{array}{c}
mn - pq + xy \\
xn + mq - py \\
-pn + xq + my \\
\hline
(m + x - p)\,n + (m + x - p)\,q + (m + x - p)\,y
\end{array}$$

$(m + x - p)$ being a common factor of the terms of this result, we have, continuing the process, $(n + q + y)(m + x - p)$ as the final answer.

Suggestion.

In adding, arrange the terms into columns according as they have the same unit of addition.

EXAMPLES.

1. Find the sum of $3a + b - 10$, $c - d - a$, $-4c + 2a - 3b - 7$ and $4x^2 + 5 - 18m$.

2. Find the sum of $7x^3 + 8b - 5$, $7b - 2x^3 + 8$, and $6 - 5x^3 + m$.

3. Find the sum of $4ax^2 + 3by^3 - c$, $4by^3 - 3ax^2 + d$ and $k - ax^2 - 7by^3$.

4. Find the sum of $7a - 5y^3$, $8\sqrt{x} + 2a$, $5y^3 - \sqrt{x}$, and $-9a + 7\sqrt{x}$.

5. Find the sum of $amx^3 + bnx^2 + cx$, $bmx^3 - anx^2 + ax$, and $cmx^3 - nx^2 + 3bx$.

6. Find the sum of $3ab + 3(a + b)$, $-ab + 2(a + b)$, $7ab - 4(a + b)$, and $-2ab + 6(a + b)$.

7. Find the sum of $6x\sqrt{a} + 5\sqrt[4]{a - b} + 7(x - y)$, $3(x - y) - 3x\sqrt{a} - 3\sqrt[4]{a - b}$, $2x\sqrt{a} - 8\sqrt[4]{a - b}$, and $4x\sqrt{a} - 5(x - y)$.

8. Find the sum of $8x^{-2}y^{-1}z^3 - 7m$, $18m - n^a m^b + x^c - 3a^5$ and $5n^a m^b - ax^{-2}y^{-1}z^3 - 3x^c + 8a^5$.

9. Find the sum of $ax + my + nz$ and $mx + ay + cd$.

10. Find the sum of $cd - ab + xy$ and $-ad + cb - my$.

11. Find the sum of $ax^2 - \sqrt{a} + 5xy - 3mn$, $2ax^2 - 6b\sqrt{a} - 3my + 5xn$, and $cx^2 + 3\sqrt{a}$.

12. Find the sum of $3my^3 - 7nx^2 - 4bc + 2y\sqrt{a}$, $5ny^3 + 2yx^2 - abc - 7n\sqrt{a}$, and $-2py^3 + 6abc + r^2$.

Positive quantities added give a positive result; negative quantities added, a negative result. This is no less true of polynomials than monomials. Thus,

$$\begin{array}{rlr}
+5 & +(a+b) & +(a+c)\sqrt{x} \\
+6 & +(a-c) & +(b-c)\sqrt{x} \\
\cline{1-2}
+11 & +(2a+b-c) & +(a+2b)\sqrt{x} \\
\cline{3-3}
 & & +(2a+3b)\sqrt{x}
\end{array}$$

and

$$\begin{array}{rlr}
-4 & -(2+c) & -(2a-b)x^2y^2 \\
-3 & -(3+a) & -(b+c)x^2y^2 \\
-2 & -(b-5) & -(c+d)x^2y^2 \\
\hline
-9 & -(c+a+b) & -(2a+2c+d)x^2y^2
\end{array}$$

EXAMPLES.

1. Find the sum of $(a-c)x - (b+d)y$, $(c-e)x - (f-d)y$, and $(e+a)x - (b-f)y$.

2. Find the sum of $(2a+3)x^2 - (3b-2a)y^2$ and $(2a-3)x^2 - (3b-3a)y^2$.

3. Find the sum of $(x-y)a^2 - (2a-c)b^2 + am$, $bm + (y-2x)a^2 - (c-2d)b^2$, $(x+z)a^2 - (3a-x)b^2 - cm$, and $(y-z)a^2 - (x-d)b^2 - dm$.

4. Find the sum of $(a-b)\sqrt{7} + (x+y)\sqrt{2} - \sqrt{x}$, $2(x+y)\sqrt{2} + (a-c)\sqrt{7} + 3\sqrt{x}$, $(b-2a)\sqrt{7} - 3(x+y)\sqrt{2}$, and $5\sqrt{x} - 8(x+y)\sqrt{2} + (2d-c)\sqrt{7}$.

5. Find the sum of $(x-y+z)\sqrt{a-b}+(a-b-c)\sqrt{x-y}$, $(y-x+z)\sqrt{a-b}+(b-a-c)\sqrt{x-y}$ and $(x-2z)\sqrt{a-b}+(3m+2c)\sqrt{x-y}$.

6. Find the sum of $(3x+y-4a)m^2-(2b-3)n^2+pz$, $(a-2x-2y)m^2-(5-3b)n^2-qz$, and $(3a+x+4y)m^2-(2-4b)n^2+m$.

SUBTRACTION.

61. **Subtraction** in algebra is the process of finding the difference between two algebraic quantities.

The quantity to be subtracted is called the *subtrahend;* the quantity from which it is to be subtracted, the *minuend;* the quantity resulting from the subtraction, the *difference* or *remainder.*

From 12 subtract $7-4$.

The difference can evidently be found by subtracting 4 from 7, and the remainder thus obtained from 12. Thus, $12-(7-4)=12-3=9$. If however 7 be first subtracted from 12, we obtain a result 5, which is *too small* by 4, since we have subtracted 4 more than was required. We therefore, in order to make the proper correction, add 4. The operation may be expressed thus, $12-7+4=9$, which it will be observed amounts to changing the signs of the subtrahend and adding the result to the minuend.

Again, from $a-b$ subtract $c+\frac{d}{e}$.

If c be first subtracted, the result will be $a-b-c$. It is plain that the amount subtracted was *too small* by the fractional quantity $\frac{d}{e}$, and that therefore the result obtained is *too large* by that amount. This can be corrected by subtracting $\frac{d}{e}$. The result will then be $a - b - c - \frac{d}{e}$.

Thus it is made clear that to subtract a quantity is the same as to add a quantity, with the opposite algebraic sign.

Hence the following

RULE FOR ALGEBRAIC SUBTRACTION.

Conceive the signs of all the terms of the subtrahend to be changed and proceed as in addition.

EXAMPLES.

1. From $4a + 3b - 2c + 8d$ subtract $a + 4b + c - 9d$.

2. From $12xy + 3y^2 - 3\sqrt{2}$ subtract $-5xy + 7y^2 + 2\sqrt{2} - 4$.

3. From $\sqrt{m-n} + 5(x^2 - y^2) - 3\sqrt{a-b}$ subtract $4(x^2 - y^2) - 2\sqrt{m-n} - 8\sqrt{a-b}$.

4. From $5z\sqrt{y-z} + 5xyz + 3abc$ subtract $12xyz - 5abc - z\sqrt{y-z}$.

5. From $cm^{-2} - dn^{-1} + ef - y$ subtract $cm^2 - dn^{-1} - zf - 2y$.

The following examples enforce the importance of a clear understanding of the meaning of an algebraic term. The student is asked to specify the terms of each subtrahend and the signs he conceives to be changed.

EXAMPLES.

1. From $(m-2)(a-b)-(n-3)(a+b)-x$ subtract $(3+n)(a+b)-(6-m)(a-b)+y$.

2. From $(x-y)mn-(a-b)(x+y)^{-8}-5x^2$ subtract $(2b+5a)(x+y)^{-8}+(m-5)x^2-mny$.

3. From $3x^2-4ay$ subtract $5x^2-\frac{c}{d}$.

4. From $4x(a+b)-2bx$ subtract $3y(a+b)+bx+\frac{c-d}{m}$.

5. From $7a(x^5-y^3)+3mn(a-h)-az$ subtract $3a(x^5-y^3)-2an(a-h)-\frac{c-(b+x)}{z}$.

After changing the signs of the subtrahend, the student should continue to combine terms until the final result is in its simplest form.

6. From $ab^2+(c-d)y$ subtract $-(c-d)b^2-ay$.

7. From $x(m-n)+y(m+n)$ subtract $-y(m-n)-x(m+n)$.

8. From $(a+b)(x-y)+(a-b)(x+y)$ subtract $-(a-b)(x-y)-(a+b)(x+y)$.

9. From $-(3a-2b)(2x+3y)-(5m+2b)(3y-2x)$ subtract $(5m+2b)(2x+3y)+(3a-2b)(3y-2x)$.

USE OF THE SYMBOLS OF AGGREGATION IN ADDITION AND SUBTRACTION.

62. Since the several symbols of aggregation employed in algebraic operations have the same significance, the principles determined as governing in the use of one will govern in the use of the others. The parenthesis is most frequently used; it is the one, therefore, taken for consideration. As already defined it indicates that the terms inclosed by it are to be treated as one quantity. We are to inquire regarding the rules for its removal and introduction when preceded by the plus or minus sign.

63. The Parenthesis preceded by the Plus Sign. The parenthesis with the plus sign before it indicates that the quantity inclosed is to be added to what precedes. As quantities can be added by simply uniting them by their signs, it is plain:

1. When the sign + precedes a parenthesis, the symbol and the sign can be omitted without affecting the value of the expression.

Thus, $6+(2+3)=6+2+3=11.$
$7+(5-2)=7+5-2=10.$
$a+(b+c)=a+b+c.$
$a+(b-c)=a+b-c.$

2. Conversely: Any number of the terms of an expression may be inclosed in a parenthesis, preceded by the sign + , without affecting the value of the expression.

Thus, $8-2+4=8+(-2+4)=+(8-2)+4.$
and, $a-b+c=a+(-b+c)=+(a-b)+c.$

64. The Parenthesis preceded by the Minus Sign. The parenthesis with the minus sign before it indicates that the quantity within the parenthesis is to be subtracted. Thus in the expression $3x - y - (a + b)$, the quantity $(a + b)$ is to be subtracted from $3x - y$.

Since the first step in subtraction is to change the signs of the terms of the quantity to be subtracted, it follows:

1. When the sign $-$ precedes a parenthesis, the symbol and its sign may be omitted if the signs of all the terms within the parenthesis are changed.

Thus, $8 - (5 + 3) = 8 - 5 - 3$.
and $a - (b + c) = a - b - c$.

2. Conversely: Any number of terms of an expression may be inclosed in a parenthesis preceded by the sign $-$ provided the signs of all the inclosed terms are changed.

Thus, $3 + 4 - 2 = 3 - (-4 + 2)$.
and $a + b - c = a - (-b + c)$.

65. Compound Parentheses. When parentheses preceded by the $+$ or $-$ sign inclose other parentheses, they may be removed by the preceding rules. The order of removal is immaterial. It is recommended to the beginner to work from the inside outward.

Thus, $a - [b - (2c + d) + 5] = a - [b - 2c - d + 5] = a - b + 2c + d - 5$.

After a little experience is gained, the student will often find it convenient to remove the outside parenthesis first.

66. Remarks on the Solution of Examples. In the application of the foregoing principles to algebraic operations, attention is called to the following:

1. The principles are limited to the parentheses of addition and subtraction. They do not affect the parentheses indicating multiplication. Thus, $a + [c + (d - e)x] = a + c + (d - e)x$, the parenthesis of multiplication being still retained.

2. It is the removal or introduction of the parenthesis, *together with its sign*, which is required. The two, parenthesis and sign, are inseparable; if the parenthesis is removed the sign goes with it; if the parenthesis is introduced the sign, expressed or understood, comes in also.

3. Because of a failure to fully appreciate this relation between the parenthesis and its sign, the latter is often confused with the sign of the first term of the inclosed quantity. In the removal, for instance, of a minus parenthesis, the minus sign is retained and attached to the first term. A continued repetition of this error is inexcusable. The sign of the parenthesis and the sign of the first inclosed term are entirely distinct and should be so held in the thought of the student. In the case cited, the former should be discarded; the latter changed.

4. In the removal or introduction of the parenthesis it is the sign of each *term* that is to be changed. Thus, the student's knowledge of a *term and its sign* is again put to the test.

5. After the removal of a parenthesis it is probable that some of the terms of the resulting expression can

be combined without again introducing the parenthesis. Unless otherwise directed, such combinations should be made. *In general*, results should be reduced to their simplest form.

6. In the introduction of a parenthesis, the accuracy of the work may be tested by removing the parenthesis and comparing the result with the original expression.

7. In future work it is not to be understood that because a parenthesis *can* be removed this *must be done*. The removal or retention and the introduction or non-introduction of the parenthesis will depend largely upon other steps to be taken and are questions upon which the student will often be called to exercise his judgment. When there are several parentheses, it is sometimes helpful to remove some, but not all of them.

67. Illustrative Examples.

Remove the parentheses in the following expressions:

1. $a-(b+c-\frac{d}{e})$.

Here the sign before the parenthesis is — and the sign of the first inclosed term + understood. Discarding *the parenthesis and its sign* and changing the sign of the inclosed terms (special attention being called to the first), we have as a result $a-b-c+\frac{d}{e}$.

2. $a-(-b-c+\frac{d}{e}-2a)$.

Proceeding as in example 1 and then collecting terms, we have $3a+b+c-\frac{d}{e}$.

3. $x - \left[\frac{d+e}{f} - (a+b)x + (c+d)(x-y) + a\right]$

The subtractive parenthesis is the only one here removed. The inclosed terms with their signs are $\frac{d+e}{f}$, $-(a+b)\ x$, $+(c+d)\ (x-y)$ and $+a$. Changing the signs of these terms as we remove the parenthesis and *its sign* we have $x - \frac{d+e}{f} + (a+b)x - (c+d)(x-y) - a$.

Inclose the last three terms of each of the following expressions in a parenthesis preceded by a — sign:

4. $2ac - 3bc - 2xy + 7d$.

Introducing the *parenthesis and its sign* and changing the signs of the terms inclosed, we have $2ac - (3bc + 2xy - 7d)$. It will be noticed that the sign of the first term inclosed, is changed from — to +, this latter sign being not written but understood. This step is distinct from the introduction of the minus sign before the parenthesis. That sign comes in with the parenthesis and is not the minus sign of the first term taken from that position and placed before the parenthesis. The minus sign of the term disappears not by such removal but by being changed to plus

5. $5xy + 7cd - 4ab + 5$.

Proceeding as in example 4, we have $5xy - (-7cd + 4ab - 5)$. Note again the distinction between the introduction of the minus sign preceding the parenthesis and the change of sign of the first term which, in this example, is from + to —.

EXAMPLES.

Remove the parentheses of addition and subtraction from the following:

1. $2x + (a - b + c)$.

2. $3m + (-5n - 2m - 7a + 3)$.

3. $7ab + (-ab - c - d)$.

4. $7a^2 - 2b^2 - (3a^2 - 5b^2 - x) - x$.

5. $5m + a - (-a + 2m - 3)$.

6. $7x + 9 - \left(x - \frac{2x - 1}{9}\right) + 7$.

7. $2mn - 3x^3 - (mn - x^3) + (x - mn) + 2x^3$.

8. $5x^4 - [3x^3 - (4x^4 - 2x^3 - 5x) + 9x^4 - 5x]$.

9. $3a^2 - \Big[-2x + (5x - a)\,b - \frac{(2a + 5)c}{7} -$

$- 2x - 3a^2) \Big]$.

10. $8ab - [3ab - (3x - a)(2b - c)(x - y)]$
$- 5ab$.

11. $a - \}2b + 3c - [4d + 3c + (2b + a)]\{$.

12. $3x - \Big\{ 5y - a(z + b) - [x - \frac{2y + z}{a + c} - 2x +$

$(5y - 2x)] \Big\}$.

13. $3z^3 - 2y^2 - \}x - z^3 - [y^2 - 2z^3 - (3y^2 - x)]$
$- 4\{- (2z^3 - 4y^2)$.

14. $-\} - 1 - [-1 - (-1) - 1] - 1\{ - 1$.

15. $b - \} a - [b - (a - b - \overline{b - a})] - b - a \{.$

16. $2x - \} 3y - [4z - (2x - \overline{3y - 4z})] \{.$

17. $-(-(-(-(-a)))) + (-(+(-a))) + a.$

18. $-\}[(-x - y) + z] - k\{.$

The following example is important because of the frequent use of the expressions involved:

19. $\begin{cases} +(+a) \\ +(-a) \\ -(+a) \\ -(-a) \end{cases}$

The student is asked to read directly the following expressions with the parentheses removed but the terms not collected.

20. $a - [a - b - (c - d)].$

21. $x - [-(y - z) + (m - \frac{x - y}{n})].$

22. $a^2 - y^2 - [m^2 - (m + 3)(n - 2) + \frac{(m - n)y}{c - d}].$

23. $x - \}y - [-z - (x - y)] + z\{.$

Without altering the value, introduce within a parenthesis, preceded by a plus sign, the following:

24. $\begin{cases} +a \\ -a \end{cases}$

25. The last three terms of $a + b - c + d.$

26. The last three terms of $x - y - z + (a - b).$

27. The last three terms of $m^2 - [a - (b - c)] + m^3 - n^2.$

Without altering the value, introduce within a parenthesis, preceded by a minus sign, the following:

28. $\left\{\begin{array}{l}+a\\-a\end{array}\right.$

29. The last three terms of $4x-3cb-8y-5+2b$.

30. The 2d, 3d, 4th, and 5th terms of $4a+3b-7c-6y+7mz-8kx$.

31. The 4th, 5th, and 6th terms of $7x-3a+2z-5\,my+(3-x)+8ab-7$.

32. The 2d and 3d terms of $9mx-3x^2+\frac{2a-1}{7}-5$.

33. The 2d and 3d, also the 5th and 6th, terms of $a-(b-c)-d+m-(x-y)\ (z-k)+y^2$.

34. The 2d and 3d terms of $m^2-[a^2-(b+c)-d^3]+(x-y)$.

35. The 2d and 3d terms of $a-[(a+b)-c]-(d-e)$.

36. The 3d and 4th terms of $a^2-b^2-(x+y)\ (m+n)-\left[x-\frac{a-b}{c-d}-k)\right]$.

CHAPTER III.

MULTIPLICATION.

68. **Multiplication** in algebra is finding the product of two algebraic quantities. These two quantities are *factors* of the product.

The one multiplied is called the *multiplicand;* the other the *multiplier.*

The product of two factors may be multiplied by a third, and this product by a fourth and so on indefinitely.

A *product* may therefore contain any number of factors.

69. The **base of a factor** is the quantity to which an exponent is attached to make the factor.

Thus, a is the base of the factor a^2.

70. **Similar factors** are those which have a common base affected by a common exponent.

Thus $a^{\frac{1}{2}}$ and $a^{\frac{1}{2}}$ are similar factors.

71. **Semi-similar factors** are those which have either a common base or bases affected by a common exponent.

Thus, 2^3 and 2^5 are semi-similar factors since they have a common base; 2^3 and 3^3 are semi-similar factors since their bases are affected by a common exponent.

72. **Unlike factors** are those which have neither a

common base nor bases affected by a common exponent.

Thus, 2^3 and 3^2 are unlike factors.

73. Factors and Terms. Care should always be taken to distinguish *factors* from *terms*. A quantity is the *product* of its *factors* and the *algebraic sum* of its *terms*.

PRELIMINARY PRINCIPLES.

74. Principle I. Factors may be taken in any order and their product remain the same.

Thus, 2×3 is the same in value as 3×2; $2 \times 3 \times 5$ the same as $5 \times 3 \times 2$ or $3 \times 5 \times 2$; abc the same as cba or bca; $4 \times -a$ the same as $-a \times 4$.

It is a convenient custom to write first the numerical factor and then the literal factors arranged in alphabetical order.

75. Principle II. To multiply any factor of a quantity multiplies the quantity.

Given the quantity $3 \times 5 \times 7$ to be multiplied by 2. It is clear that the result must be equal in value to 210.

Multiplying the first factor by 2 gives $6 \times 5 \times 7 = 210$.

Multiplying the second factor by 2 gives $3 \times 10 \times 7 = 210$.

Multiplying the third factor by 2 gives $3 \times 5 \times 14 = 210$

Given the quantity abc to be multiplied by d, in which a, b, c, and d are conceived as having any value.

Multiplicand × multiplier $= abc \times d = abcd$.

First factor multiplied by d gives $\overline{ad}bc = abcd$.

Second factor multiplied by d gives $a\overline{bd}c = abcd$.

Third factor multiplied by d gives $ab\overline{cd} = abcd$.

Thus multiplying any factor by d gives the correct product.

Hence the principle, which evidently holds when the quantity multiplied has *any number* of factors of *any value*.

76. Principle III. To multiply by any number of factors in succession gives the same result as to multiply at once by their product.

Thus, to multiply a quantity by 2 and this result by 3 is the same as to multiply the quantity by 6; or desiring to multiply by 30 we can multiply successively by 2, 3, and 5. Multiplying by a, b, c, etc., in any order is the same as multiplying by abc, etc.

77. Combining Principle III with Principle II, it is clear that when the multiplicand and multiplier are composed of factors, these factors can, when convenient, be multiplied together in pairs.

Thus,

$$\begin{array}{lll} \text{multiplicand} & 3 \times 2 \times 6 & = 36 \\ \text{multiplier} & \underline{2 \times 4 \times 5} & \underline{= 40} \\ \text{product} & 6 \times 8 \times 30 & = 1440 \end{array}$$

Again,

$$\begin{array}{lll} \text{multiplicand} & a \times b \times c & = abc \\ \text{multiplier} & \underline{d \times e \times f} & \underline{= def} \\ \text{product} & ad \times be \times cf & = abcdef \end{array}$$

78. Principle IV. Multiplying all the terms of a quantity multiplies the quantity.

Thus,

$$\begin{array}{lrr} \text{multiplicand} & 2+3+5 & =10 \\ \text{multiplier} & 3 & =\ \ 3 \\ \hline \text{product} & 6+9+15 & =30 \end{array}$$

Again,

$$(x+y+z)\,2=(x+y+z)+(x+y+z)$$
$$=2x+2y+2z.$$

and

$$(x+y+z)\,m=(x+y+z)+(x+y+z)$$
$$\ldots\text{to } m \text{ terms.}$$

Collecting terms $= mx+my+mz$.

Since m may be considered as representing any quantity, monomial or polynomial, it is plain that

$$(x+y+z)\,(a+b)=x\,(a+b)$$
$$+y\,(a+b)+z\,(a+b).$$

LAW OF SIGNS.

79. *The product of two factors is positive when they have like signs, and negative when they have unlike signs.*

CASE I. $+\times+$. Let the factors be $+a$ and $+3$. The multiplier tells us that the multiplicand $+a$ is to be added 3 times. This will evidently give as a result $+a+a+a=+3a$. If the multiplier were $+b$, the product would still be $+$, since a positive quantity *added* any number of times will clearly give a positive result.

Case II. $- \times +$. Take $-a$ and $+b$ as factors. In common language this means that a minus quantity is to be *added* a given number of times. The sum will evidently be —.

Case III. $+ \times -$. Take the factors a and $-b$. Applying the principle that "the product of two factors is the same in whatever order they are taken", $a \times -b = -b \times a$. The sign of the product is then determined by Case II, and is clearly —.

Case IV. $- \times -$. Given $-a$ and $-b$ as factors. The multiplier in this case is the opposite of that in Case II. There $-a$ is multiplied by b, i. e., $-a$ is *added* b times. Here $-a$ is multiplied by $-b$, i. e., $-a$ is *subtracted* b times, or what is the same thing, $-a$ is taken b times and then this sum subtracted. In either case we obtain a positive result. The two methods of operation may be expressed algebraically thus,

$$-a \times -b = -(-a) \times b = a \times b = ab.$$
$$-a \times -b = -(-a \times b) = -(-ab) = ab.$$

Summarizing the four cases we have:

$$+ \times + = +$$
$$- \times + = -$$
$$+ \times - = -$$
$$- \times - = +$$

which proves the *law of signs*. The statement of this law is often condensed into the form:

Like signs produce + and unlike signs produce —.

If the product consists of several factors, the sign of each factor must be taken into account. It is plain that any number of + factors multiplied together will give a + product, while an even number of — factors will give a + product and an odd number of — factors a — product.

The sign of any two factors can then be changed without changing the value of the product.

RULES OF OPERATION.

80. Multiplication of Monomials. The product of any number of monomials may be expressed by simply writing them as factors. Thus,

$$ab \times cd = abcd.$$

$$3ab \times 2c = 3ab2c.$$

This latter result may be put in simpler form by combining the numerical factors. Thus,

$$3ab2c = 3 \times 2abc = 6abc.$$

Combinations of literal factors also can often be made. In general, factors may be combined or multiplied together in the two following cases:

CASE 1. *When the factors have the same base.*

CASE 2. *When the bases of the factors are affected by the same exponent.*

These two cases give rise to two rules for the multiplication of monomial factors.

FIRST RULE.

When the factors have the same base, attach to that base as an exponent the sum of the exponents of the common base.

Required to multiply a^3 by a^2.

As shown by the exponents, the base a is taken three times as a factor in the multiplicand and two times in the multiplier. It must, therefore, appear five times as a factor in the product.

Thus,

$$\begin{array}{ll} a^3 & = aaa. \\ \underline{a^2} & = \underline{aa.} \\ a^5 & = aaaaa. \end{array}$$

Required to multiply a^m by a^n.

Here the base a appears m times as a factor in the multiplicand and n times in the multiplier; consequently $m + n$ times in the product.

Therefore $a^m \times a^n = a^{m+n}$. Hence the rule.

Solution of examples. In the solution of examples, under either rule for the multiplication of monomials, the student should stand ready always:

1. To state which factors can be combined and why.

2. To name the base of each factor.

3. To give the exponent of each base.

4. To give the rule employed in the multiplication.

5. To explain how the sign of the product is determined.

EXAMPLES.

1. $a^3 \times a^5$ $\qquad a \times a^m$ $\qquad a^{\frac{1}{3}} \times a^{-\frac{2}{3}}$.

$a^5 \times a^2$ $\qquad a^{-n} \times a^{-m}$ $\qquad a^{-\frac{1}{3}} \times a^{-\frac{2}{3}}$.

$a^5 \times a^{-7}$ $\qquad a^{\frac{1}{2}} \times a^{\frac{1}{4}}$ $\qquad a \times a^{\frac{1}{3}}$.

2. $a^p b^q \times a^m b^n c^r$.

3. $a^{-3} b^{-2} \times - a^{-9} b^4$.

4. $- a^{p-q} b \times a^{3q-2} c x$.

5. $a^{-m} b^p \times a^n b^{-r} \times a^{n+m} b$.

6. $- y^{\frac{1}{y}} b^m \times y^{\frac{5}{y}} b^n$

7. $2^z \times 2^{y-z}$.

8. $6^{-3} \times 6^{-\frac{1}{2}}$.

9. $(a + y)^{-3h^5l^4} \times (a + y)^{m+3l-4m}$.

10. $y^2 (x + \sqrt{2})^{-3} \times y^{-3} (x + \sqrt{2})^{-2}$.

11. $(\sqrt{x - a} + 1)^m \times (\sqrt{x - a} + 1)$.

SECOND RULE.

When the bases of the factors have a common exponent, multiply the bases together and to the product attach the common exponent.

Required to multiply 3^2 by 2^2.

$$3^2 = 3 \times 3$$
$$2^2 = 2 \times 2$$

Multiplying the factors by pairs we have 6×6
$$= 6^2$$

which it will be noted equals 36 or the same as would have been obtained if each factor had been expanded and then the two multiplied together.

Required to multiply a^n by b^n.

Here a appears n times as a factor in the multiplicand and b, n times as a factor in the multiplier. Conceiving both multiplicand and multiplier to be expanded and the factors of the two multiplied together in pairs, we have ab taken n times as a factor in the product.

Therefore $a^n \times b^n = (ab)^n$.

This may be put in algebraic form thus,

$$a^n \times b^n = (a \times a \times a \times a \ldots\ldots\ldots\ldots \text{taken } n \text{ times})$$
$$(b \times b \times b \times b \ldots\ldots\ldots\ldots \text{taken } n \text{ times})$$

Multiplying factors by pairs. $\Big\} = (ab) \times (ab) \times (ab).\text{taken } n \text{ times}$

$$= (ab)^n.$$

EXAMPLES.

1. $2^3 \times 5^3$. Ans. 10^3 or 1000.
2. $2^{-3} \times 6^{-3}$.
3. $5^{\frac{1}{2}} \times 7^{\frac{1}{2}}$.
4. $(x - y)^{-2} \times x^{-2}$ Ans. $(x^2 - xy)^{-2}$.
5. $(b + c + e)^m \times a^m$.
6. $(x^3 - y^3)^{\frac{1}{2}} \times x^{\frac{1}{2}}$.
7. $6^{-1}a^{-2}(x - y)^n \times 2^{-1}(c - d)^{-2}x^n$.

Other examples involving the use of the foregoing rules will be given after the multiplication of polynomials has been considered.

81. Multiplication of Polynomial Factors.

RULE.

Multiply each term of the multiplicand by each term of the multiplier and add the partial products.

Multiply $(a+b)$ by $(c+d)$.

By Principle I, $(a+b)(c+d)=(c+d)(a+b)$

By Principle IV, $=c(a+b)+d(a+b)$

By Principle I, $=(a+b)c+(a+b)d$

By Principle IV, $=ac+bc+ad+bd$

which is each term of the multiplicand, $a+b$, multiplied by each term of the multiplier, $c+d$, and the partial products then added. The same reasoning will apply when the multiplicand and multiplier are composed of any number of terms. Hence the rule.

EXERCISES.

1. Give demonstration of the rule when the multiplicand contains three terms and the multiplier two.

2. Give demonstration of the rule when the multiplicand and multiplier each contain three terms.

EXAMPLES.

Multiply the following:

1. $2x+3y$ and $x+y$.
2. a^2+ab+b^2 and $a-b$.

3. $3a + 2bx - 3x^2$ and $3a - 2bx + 3x^2$.

4. $ab + a^2 + b^2$ and $a^2 - ab + b^2$ and $a^2 - b^2$.

5. $11m^{-5}n^3 - 6m^{-4}n^2 + 2m^{-3}n$ and $8m^{-4}n^2 + m^{-3}n$.

6. $x^{2y} - 2x^yz + z^2$ and $x^y - z$.

7. $x^{m+n-1} - x^{m+n-2} + x^{m+n-3}$ and $x^2 + x$.

8. $x^{2p+2} - x^{p+1}y^q + y^{2q}$ and $x^{2p+2} + x^{p+1}y^q + y^{2q}$

82. Remarks on Miscellaneous Examples.

1. In multiplying together similar factors either rule for the multiplication of monomials may be used.

Thus,

By first rule $2^2 \times 2^2 = 2^4$ (expanded) $= 16$.

By second rule $2^2 \times 2^2 = 4^2$ (expanded) $= 16$.

2. Unlike factors can often to advantage be reduced to semi-similar factors. The proper rule may then be applied.

Thus,

$$\text{Multiply } \left(\frac{4}{5}\right)^7 \text{ by } \left(\frac{5}{8}\right)^6.$$

In this form the factors have neither a common base nor bases affected by a common exponent. The example can however be written $\left(\frac{4}{5}\right) \times \left(\frac{4}{5}\right)^6 \times \left(\frac{5}{8}\right)^6$. The bases of the last two factors have now a common exponent and applying the second rule we obtain $\frac{4}{5} \times \left(\frac{1}{2}\right)^6$. Expanding $\left(\frac{1}{2}\right)^6$, the example will again come under the second rule, being $\frac{4}{5} \times \frac{1}{64}$ in which the common exponent is 1, the final product being $\frac{1}{80}$.

3. When unlike factors cannot be thus reduced their multiplication can only be expressed.

Thus,

$$a^m \times b^n = a^m b^n.$$

4. In the process of multiplication the student should work from the left to the right.

5. In the multiplication of polynomials, it will generally be found convenient to arrange the terms of each polynomial in the order of the ascending or descending powers of some common letter.

6. The addition of partial products should never be neglected. This process is called *collecting the terms.*

7. It is sometimes better to indicate the multiplication than to actually perform the operation. This is well illustrated in certain of the following examples. A similar principle was also made clear in the examples in addition.

MISCELLANEOUS EXAMPLES.

Multiply the following:

1. $5(x+y)b^{-2}c^2$ and $4(x+y)^{-n}b^nc$.

2. $3(x+y)^{m+3}b^nc^2$ and $-7(x-y)^{m+3}bc^{-4}$.

3. $-9bc^{-2}(\sqrt{x-y}+\sqrt{x+y})^n$ and $-8b^{-n}c^3(\sqrt{x-y}+\sqrt{x+y})^3$.

4. $8b^3c^x(x-y)^4$ and $7b^mc(x-y)^4$.

5. $-6x^3(a-b)^3(x-y)^m$ and $4x^r(a+b)^3(x-y)$.

6. $-3k^{-4}l^n(a-b)^n(c+d)^{-x}$ and $5k^nl^{-3}(a+b)^n(c+d)^{-x}$.

7. $-8a^2b^{-1}(x-a)^{\frac{1}{3}}(c+d)^{\frac{1}{5}}$ and $-4a^{-5}b^{-n}(x-a)^{\frac{1}{2}}(c-d)^{\frac{1}{5}}$.

8. $15a^{-3}k^3(m+n)^m$ and $-7a^3k^{-4}(m-n)^n$.

9. $12m^{\frac{1}{2}}c^2(a+b)^{\frac{1}{m}}(x-y)^{\frac{1}{n}}$ and $7mc^n(a+b)(x+y)^{\frac{1}{n}}$.

10. $-6a^{\frac{1}{3}}b^{\frac{1}{7}}(a+m)^{\frac{1}{9}}$ and $-3a^{-\frac{1}{6}}b^{\frac{1}{2}}(b-n)^{\frac{1}{9}}$

11. $-5ab^{\frac{1}{2}}c^y(x-2\sqrt{y})^3$ and $8a^{-\frac{1}{2}}bc^n(x-2\sqrt{y})^{-\frac{1}{2}}$.

12. 2^8 and 4.

13. $\left(\frac{2}{3}\right)^4$ and $\left(\frac{3}{4}\right)^6$.

14. $\left(\frac{3}{5}\right)^6$, $\left(\frac{5}{6}\right)^5$ and $\left(\frac{2}{3}\right)^4$.

15. $6x^4-3x^3-x^2+6x-2$ and $2x^2+x+2$.

16. $m^2+n^2+r^2-mn-nr-mr$ and $m+n+r$.

17. $a+b$, $a-b$, $a-2b$, and $a+2b$.

18. m^n+x^y and m^n-x^y.

19. $x^{-5}-x^{-4}+x^{-3}-x^{-2}+x^{-1}$ and $x^{-1}+1$.

20. $x^{y+2} - x^{y+1} + x^{y} - x^{y-1} + x^{y-2}$ and $x + 1$.

21. $a^{x-2}y^{c+1} + a^{x-3}y^{c+2} + a^{x-4}y^{c+3}$ and $a^{x+1}y^{c} - a^{x}y^{c+1}$.

Remove the parentheses of multiplication from the following:

22. $c\,(x^2 + cx + c^2)$.

23. $(a^2 + 2a + 4)\,(a^2 - 2a + 4)$.

24. $(m^{\frac{1}{2}} - 5)\,(-\,m^{\frac{1}{2}} - 4)$.

25. $(a^{\frac{1}{3}} + 3)\,(a^{\frac{1}{3}} + 4)\,(a^{\frac{1}{3}} + 5)$.

26. $\left(\frac{3}{5}a^{\frac{3}{5}}\,b^{-\frac{1}{2}} + \frac{2}{3}x^{\frac{2}{3}}\,y^{-\frac{1}{3}}\right)\left(\frac{3}{5}a^{\frac{3}{5}}\,b^{-\frac{1}{2}} - \frac{2}{3}x^{\frac{2}{3}}\,y^{-\frac{1}{3}}\right)$.

Remove all the parentheses in the following:

27. $x^2 + y^2 - (x - y)\,(x + y)$.

28. $2m^3 + 6n^3 - (m - n)\,(m + 2n)\,(2m - 3n) - mn\,(m - n)$.

29. $\frac{1}{2}\,[4a - 6\,(a - b)] - \frac{1}{3}\left\{9a - [6b - (-\,3a - 12b)]\right\}$.

30. $9\left\{3 - \frac{1}{9}\,[6 - 3\,(4 - \frac{1}{3})] - 6\right\}$.

82. Special Rules in Multiplication.

The results of certain multiplications are so important as to require special mention. In the algebraic form, in which they first appear they are called *formulas*; translated into common language, they are *rules of operation*.

(a) Multiply $a + b$ by $a + b$.

Operation,

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \qquad\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

Therefore $(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$.

Since a and b may each represent quantities containing any number of terms either simple or complex, $a + b$ may be considered as *the sum of any two quantities.* Expressing in common language the result of the multiplication of $a + b$ by $a + b$ we have as the

First Special Rule. *The square of the sum of any two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

EXAMPLES.

Determine by inspection the values of the following:

1. $(x + 2y)(x + 2y)$.
2. $(3m + 5n)(3m + 5n)$.
3. $\left(4a + \frac{1}{2}\right)\left(4a + \frac{1}{2}\right)$.
4. $(2x^2 + 3a)(2x^2 + 3a)$.
5. $(x^m + 2ax^n)(x^m + 2ax^n)$.

6. $(a^{-5}+5b^{-3})(a^{-5}+5b^{-3})$.

7. $(3a^{-3}c+2bc^2)(3a^{-3}c+2bc^2)$.

8. $[2a+(x+y)][2a+(x+y)]$.

(b) Multiply $a-b$ by $a-b$.

Operation,

$$\begin{array}{lll} a & -b & \\ a & -\mathrm{b} & \\ \hline a^2 & -ab & \\ & -ab & +b^2 \\ \hline a^2 & -2ab & +b2 \end{array}$$

Therefore $(a-b)^2=a^2-2ab+b^2$.

Translated into common language this gives us the

Second Special Rule. *The square of the difference of any two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

EXAMPLES.

Determine by inspection the value of the following:

1. $(m-3n)(m-3n)$.

2. $(4x-3y)(4x-3y)$.

3. $\left(\frac{2}{3}a-\frac{1}{5}y\right)\left(\frac{2}{3}a-\frac{1}{5}y\right)$.

4. $(x^2-3ax)(x^2-3ax)$.

5. $(y^{-n}-x^m)(y^{-n}-x^m)$.

6. $(2abc^{-1}-3a^2)(2abc^{-1}-3a^2)$.

7. $(4x^{-2}-3a^{-1}y^{-2})\ (4x^{-2}-3a^{-1}y^{-2})$.

8. $[(x-y)-z]\ [(x-y)-z]$.

(c) Multiply $a+b$ by $a-b$.

Operation,

$$\begin{array}{lcr} a+b & & \\ a-b & & \\ \hline a^2+ab & & \\ \quad\ -ab & + & b^2 \\ \hline a^2 & & -b^2 \end{array}$$

Therefore $(a+b)\ (a-b)=a^2-b^2$, which equation or *formula* translated gives the

Third Special Rule. *The product of the sum of any two quantities by their difference is equal to the difference of their squares.*

EXAMPLES.

Determine by inspection the values of the following:

1. $(a-2c)\ (a+2c)$.

2. $(2x-3y)\ (2x+3y)$.

3. $\left(\frac{1}{2}m-\frac{1}{3}n\right)\left(\frac{1}{2}m+\frac{1}{3}n\right)$.

4. $(p^2-q^3)\ (p^2+q^3)$.

5. $(ax^{-1}+by^{-2})\ (ax^{-1}-by^{-2})$.

6. $(3xy^{-3}+4az^{-1})\ (3xy^{-3}-4az^{-1})$.

7. $(2m^{-x}y^{z}+5b)\ (2m^{-x}y^{z}-5b)$.

8. $[(a-2b)-c]\ [(a-2b)+c]$.

It is to be noted in the application of this rule that while one of the two factors to be multiplied together is the sum of *any two quantities*, the other factor is the *difference of the same two quantities.* A due appreciation of this fact will save the beginner from many annoying blunders.

(d) Multiply $x + a$ by $x + b$.

Operation,

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \qquad\quad bx + ab. \\ \hline x^2 + (a + b)\,x + ab. \end{array}$$

Therefore $(x + a)\,(x + b) = x^2 + (a + b)\,x + ab$

Similarly $(x + a)\,(x - b) = x^2 + (a - b)\,x - ab$

$(x - a)\,(x - b) = x^2 + (- a - b)\,x + ab$

$(x - a)\,(x + b) = x^2 + (- a + b)\,x - ab$

These results interpreted give the

Fourth Special Rule. *The product of any two binomials whose first terms are the same, is equal to the square of the first term, plus the algebraic sum of the second terms into the first, plus the algebraic product of the second terms.*

When the second term of the result contains a polynomial factor such factor must of course be inclosed in a parenthesis. The sign of the term should be made minus when necessary to make the sign of the first term within the parenthesis plus. Any numerical factor should be written first in order.

Thus,

$$\begin{aligned}(2a-b)(2a-c) &= 4a^2+(-b-c)\,2a+bc\\ &= 4a^2-2(b+c)\,a+bc.\end{aligned}$$

EXAMPLES.

Determine by inspection the product of the following:

1. $(x+3)(x+4)$.
2. $(y-7)(y-5)$.
3. $(z^2+9)(z^2-4)$.
4. $(a^2+1)(a^2-3)$.
5. $(x-3a)(x+5a)$.
6. $(y+cx)(y-bx)$.
7. $(ab-7c)(ab+3c)$.
8. $(2x-5)(2x-4)$.
9. $(3a^2+2c)(3a^2-7c)$.
10. $(2x^{\frac{1}{2}}+1)(2x^{\frac{1}{2}}-13)$.
11. $(5x^m-3)(5x^m+2)$.
12. $(a^{\frac{3}{4}}+1)\left(a^{\frac{3}{4}}-\frac{1}{2}\right)$.
13. $(2a^{-m}x^{-1}-3)(2a^{-m}x^{-1}-5)$.
14. $[(a-b)-3]\,[(a-b)+7]$.

(e) Multiply $a+b$ by $c+d$.

Operation,

$$\begin{array}{l} \quad a + b \\ \quad c + d \\ \hline ca + cb \\ \qquad\qquad + da + db \\ \hline ca + cb + da + db \end{array}$$

Therefore $(a + b)(c + d) = ca + cb + da + db$.

Designating for convenience,

a and c as *first terms* since they are first terms of the binomials,

b and c as *middle terms* because of their position,

a and d as *extreme terms* because of their position,

b and d as *second terms* since they are second terms of the binomials, we have the

Fifth Special Rule. *The product of any two binomials written in the form* $(a + b)(c + d)$ *is equal to the algebraic sum of the products of the first terms, the middle terms, the extreme terms and the second terms.*

Thus,

$$(3a^2b - 2ac)(2ad + 3ab) = 6a^3bd - 4a^2cd + 9a^3b^2 - 6a^2bc.$$

EXAMPLES.

Determine by inspection the product of the following:

1. $(a + 5)(b + 6)$.

2. $(x - 2)(y - 4)$.

3. $(m + 2)(n - 3)$.

4. $(2a + c)(5b - d)$.

5. $(5a^2m - 4xy)(3am^2 - 2ax)$.

6. $(2by^2 - 3ax^{-1})(4cy + 7ax^{-2})$.

7. $(5x^{-1}y^{-2} - 2xy^2)(2x^2y + 5x^{-2}y^{-1})$.

8. $[(a - b) - 5][(a + b) + 6]$.

As the ability to state, at once, the product of any two binomials, without performing the actual multiplication, is exceedingly helpful in algebraic work, the following practice examples are given. The student is required in each example,

1. To give the number of the special rule used.
2. To give the rule.
3. To state the product.

PRACTICE EXAMPLES.

1. $(3x - 7y)(3x + 7y)$.
2. $(ax + bz)(ax + bz)$.
3. $(3my - 4n)(3my + 5n)$.
4. $(7xy - 2a)(7xy - 2a)$.
5. $(ax + by)(cz - dk)$.
6. $(ax^{-1} + by^{-2})(ax^{-1} + by^{-2})$.

7. $(2y^{-3}-3a)(2y^{-3}+5a)$.
8. $(x^{-1}+2y^{-2})(3z^{-3}-4k^{-4})$.
9. $(3mn^{-4}-7pq)(3mn^{-4}+7pq)$.
10. $(5m^{n}-3x^{y})(5m^{n}+4x^{y})$.
11. $(2ab^{c}+4)(3bc^{d}-5)$.
12. $(3a-6z^{x})(3a-6z^{x})$.
13. $(5m^{-1}y-2b^{a}c^{-3})(5m^{-1}y+2b^{a}c^{-3})$.
14. $[(a+x)-y][(a+x)+y]$.
15. $[(2a-b)+n][(2a-b)+n]$.
16. $[(3a+4y)+6][(3a+4y)-7]$.
17. $[(a+2b)-3c][(a-2b)-2c]$.
18. $[3a-(x-y)][3a-(x-y)]$.

It is sometimes convenient to arrange the terms of the quantities in a different order before multiplying.

19. $(m-n+c)(m+n+c)$
20. $(x^2-xy+y^2)(x^2+xy+y^2)$.
21. $(a^2+ab+b^2)(a^2-2ab+b^2)$.

Form of the Product of Binomials.

Products of binomials, obtained under the foregoing special rules, are often changed in form by performing one or more of the following operations:

1. Removing parentheses.
2. Collecting terms.
3. Changing the order of the terms.

Thus,

$$[(a+b)+c]\,[(a+b)-c]=(a+b)^2-c^2,$$
(removing the parenthesis) $= a^2+2ab+b^2-c^2,$
(changing the order of terms) $= a^2-c^2+b^2+2ab.$

The reverse operation of changing the product, given in any form, into the form first obtained under the rule of multiplication employed, is in general more difficult. The steps necessary to be taken can, however, be often discovered by inspection. They are chiefly in the line of,

1. Changing the order of the terms.
2. Introducing suitable parentheses.

A few examples are given. They are intended to test the student's grasp of the special rules and call his attention to their use in work which will be found hereafter to be exceedingly important.

EXAMPLES.

Determine by inspection the rule under which each of the following products was obtained and put each product in the form resulting from the direct application of the rule.

1. $12ab+4a^2+9b^2$.
2. $16y^2+4x^2-16xy$.
3. $a^2-c^2-b^2-2bc$.
4. $a^2+c^2-b^2-2ac$.

5. $a^2 - c^2 - b^2 + 2bc$.

6. $x^2 - c^2 + y^2 + 2cd + 2xy - d^2$.

7. $2c(a - b) + c^2 + a^2 + b^2 - 2ab$.

8. $a^2 - 2xy + x^2 - 2ax + y^2 + 2ay$.

9. $x^2 + ab + ax + bx$.

10. $x^2 + ab - ax - bx$.

CHAPTER IV.

DIVISION.

83. Division is the converse of multiplication. The product and one of the factors are given, to find the other factor. The product is called the *dividend*, the given factor the *divisor*, and the other factor the *quotient*.

The converse of principles II, III, and IV in multiplication give the following preliminary principles in division.

PRELIMINARY PRINCIPLES.

84. Principle I. To divide any factor of a quantity divides the quantity.

Thus, the quantity $2 \times 4 \times 6 = 48$ can be divided by 2, by dividing any one of the factors by 2, giving in each case a result equal in value to 24.

85. Principle II. To divide by any number of factors in succession is the same as to divide at once by their product.

Thus, to divide successively by 2, 3, and 5, taken in any order, is the same as to divide by their product, 30.

It is clear from this principle that striking out or *cancelling* factors is the same as dividing by their product.

Thus, to strike the factors a and b from abc is the same as to divide that quantity by ab.

86. Principle III. To divide all the terms of a quantity divides the quantity.

Thus,

$$\begin{array}{r} 3\underline{)\,3+6+9} \\ 1+2+3 \end{array} \text{ or } \begin{array}{r} 3\underline{)\,18} \\ 6 \end{array}$$

$$\begin{array}{r} b\underline{)\,ab+bc+bd} \\ a+c\;+\;d \end{array}$$

It is evident from this principle that striking out or *cancelling* the same factor from all the terms of a polynomial is dividing the polynomial by this factor. Cancelling the factor from *one* term of a polynomial is clearly not dividing the polynomial.

LAW OF SIGNS.

87. *The quotient is positive when the dividend and divisor have like signs and negative when they have unlike signs.*

In multiplication it was proven that,

$$\begin{array}{c} + \times + = + \\ + \times - = - \\ - \times + = - \\ - \times - = + \end{array}$$

from which it is evident that,

$$\begin{array}{c} + \div + = + \\ - \div - = \\ - \div + = - \\ + \div - = - \end{array}$$

Hence the law of signs.

RULES OF OPERATION.

88. Division of Monomials.

The first two preliminary principles make clear the steps to be taken in dividing one monomial by another. Each factor of the divisor must, in its turn, be divided into some one (the most convenient) factor of the dividend.

Thus dividing $8a^3b^2c^2$ by $2abc$, the first factor of the divisor is divided into the first factor of the dividend, the second factor of the divisor into the second factor of the dividend and so on.

Division of monomials is thus seen to consist of a repetition of the process of dividing a factor of the dividend by a factor of the divisor.

In this partial division two cases arise, which are the converse of those in multiplication.

Case I. When the dividend factor and the divisor factor have the same base.

Case II. When the bases of the dividend factor and the divisor factor have the same exponent.

These cases give rise to two corresponding rules.

FIRST RULE.

When the dividend and the divisor factors have the same base, give to the common base an exponent equal to its exponent in the first minus its exponent in the second.

This rule is the converse of the first rule in multiplication.

Remarks on Examples. In explaining examples in division of monomials, the student should be prepared to give:

1. The dividend and the divisor factors used in the partial division.
2. The base of each.
3. The exponent of each base.
4. The rule employed in the division.
5. The law for determining the sign of the quotient.

EXAMPLES.

Divide the following:

1. a^8 by a^6 a^m by a^{-n} a by $a^{\frac{1}{3}}$
   a^2 by a^8 a by a^m $a^{\frac{1}{6}}$ by $a^{\frac{1}{4}}$
   a^{-5} by a^{-7} a^{-x} by a^{-n} $a^{-\frac{1}{5}}$ by $a^{-\frac{1}{10}}$.
2. a^mb^nc by $a^xb^{-y}c^r$.
3. $a^{-7}b^{-3}$ by $-a^{-8}b^3$.
4. $-a^{p-q}b^4cx$ by $a^{3q}b^{-2}cx^{-1}$.
5. $-x^{\frac{4}{x}}y^rc^{\frac{2}{3}}b$ by $x^{\frac{3}{x}}b^{-n}c^{-\frac{1}{2}}y^{-5}$.
6. 3^m by 3^{m-n}.
7. $6^{\frac{1}{3}}$ by $6^{-\frac{1}{3}}$.
8. $(a-x)^{-5}h^7r^3m$ by $(a-x)^{5-m}r^{-1}m^{-3}$.

9. $x^3(y-\sqrt{a})^{-8}$ by $x^{-3}(y-\sqrt{a})^{-6}$.

10. $(\sqrt{a-b}-7)$ by $(\sqrt{a-b}-7)^n$.

11. $(a+x)^{-\frac{3}{4}}xy$ by $(a+x)^{\frac{9}{4}}x^{-4}y^5$.

Special Principle I. Any quantity with an exponent 0 is equal to 1.

$a^n \div a^n = a^0$, by the law of exponents.

$a^n \div a^n = 1$. A quantity divided by itself equals 1.

$\therefore a^0 = 1$.

a represents a quantity composed of any number of factors or any number of terms.

Thus,

$$2^0 = 1.$$

$$(ab)^0 = 1.$$

$$(a+b)^0 = 1.$$

$$(ab+c^2+x^n)^0 = 1.$$

It is evident that any factor having an exponent 0 can be introduced into a quantity without altering its value, for by this corollary such factor equals 1.

Special Principle II. A factor may be transferred from the numerator to the denominator of a fraction and vice versa by changing the sign of its exponent.

$$a^3 \div a^5 = a^{-2}.$$

$$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}.$$

$$\therefore a^{-2} = \frac{1}{a^2}.$$

Again,

$$\frac{a^n}{b^n} = \frac{a^n b^0}{b^n} = a^n b^{-n}.$$

In the application of this principle it should be especially noted:

1. That *factors* not *terms* are transferred.

2. That such transfer does not change a *factor* into a *term*.

EXAMPLES.

Without changing the value of the fractions, transfer factors as indicated in the following:

(a) From numerator to denominator.

1. ab^{-3} in $\dfrac{ab^{-3}c}{de^4f}$. Ans. $\dfrac{c}{a^{-1}b^3de^4f}$

2. All factors in $\dfrac{a^2n^{-3}b^n}{x^2ny}$.

3. All factors in $\dfrac{x^2(a-b)^{-3}}{y^2z^2(a+b)^3}$.

(b) From denominator to numerator.

4. $a^{-3}x^3$ in $\dfrac{m^2n^3y}{a^{-3}x^3c^{-2}}$.

5. All factors in $\dfrac{m^{-1}n^2x^{-n}}{a^{-5}b^4c^{-2}}$.

6. All factors in $\dfrac{(x-a)^4(x+y)^{-3}}{(x+a)^{-3}(x+y)^3}$.

Free the following from negative exponents:

7. $\dfrac{5cd^{-1}bx^{-4}}{8a^{-2}x^2y^{-1}z}$.

8. $\dfrac{2^{-1}a^{-\frac{1}{2}}x^{-\frac{2}{3}}y}{3m^2n^{-1}x^2}$.

9. $\dfrac{7a^3(x-y)^{-2}(x+a)^3}{3a^{-2}(x+y)^2(x+a)^{-3}}$.

SECOND RULE.

When the bases of the dividend and divisor factors have the same exponent, divide the base of the first by the base of the second and to the quotient attach the common exponent.

This rule is the converse of the second rule in multiplication.

EXAMPLES.

Divide the following:

1. 12^5 by 4^5. Ans. 3^5 or 243.

2. 81^{-7} by 3^{-7}.

3. $45^{\frac{1}{3}}$ by $5^{\frac{1}{3}}$.

4. $(xz-z^2)^{-2}$ by z^{-2}. Ans. $(x-z)^{-2}$.

5. $(ax+ay+az)^m$ by a^m.

6. $(m^5-m^4y+my^2)^{\frac{1}{5}}$ by $m^{\frac{1}{5}}$.

7. $12^{-1}(x^2+mx)^n(xy-ny)^{\frac{1}{x}}$ by $2^{-1}x^ny^{\frac{1}{x}}$.

89. Division of one Polynomial by another.

The dividend being the product of the divisor and quotient is evidently the sum of the partial products obtained by multiplying each term of the one by each term of the other. These partial products will, in some cases, have been combined in the process of collecting terms, but it is clear that the term of the dividend which is the product of the terms in the divisor and quotient, in which are found the highest powers of some particular letter, will contain that letter to a higher degree than appears in the other terms, and thus will not have been united with them. It follows then that if the terms of the dividend and divisor are first arranged according to the descending powers of this particular letter, the first term of the *quotient* can be obtained by dividing the first term of the dividend by the first term of the divisor.

If the divisor as a whole be then multiplied by this term of the quotient and the product subtracted from the dividend, a remainder will be obtained which is evidently the product of the divisor by each of the *remaining* terms of the quotient. Dividing the first term of this remainder, which will necessarily contain the highest power of the letter according to which the terms are arranged, by the first term of the divisor, will clearly give us the second term of the quotient, and so on.

Hence the following

RULE.

1. *Arrange the dividend and divisor in the order of the descending powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor and the result will be the first term of the quotient.*

3. *Multiply the whole divisor by this term and subtract the product from the dividend.*

4. *Regard this remainder as a new dividend and proceed as before, continuing until there is no remainder or one whose first term is not divisible by the first term of the divisor.*

5. *In case there is a remainder, such as last mentioned, place it over the divisor, in the form of a fraction, and add it to the quotient already obtained.*

EXAMPLES.

Divide the following:

1. $a^4 + 6a^2x^2 - 4a^3x + x^4 - 4ax^3$ by $a^2 - 2ax + x^2$.

$$\begin{array}{r l}
a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 & (a^2 - 2ax + x^2 \\
\cline{2-2}
a^4 - 2a^3x + a^2x^2 \qquad\qquad\qquad\quad & a^2 - 2ax + x^2 \\
\cline{1-1}
-2a^3x + 5a^2x^2 - 4ax^3 + x^4 & \\
-2a^3x + 4a^2x^2 - 2ax^3 \qquad & \\
\cline{1-1}
a^2x^2 - 2ax^3 + x^4 & \\
a^2x^2 - 2ax^3 + x^4 & \\
\cline{1-1}
\end{array}$$

2. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.

3. $a^4 - b^4$ by $a - b$.

4. $x^5 + x^4 - 8x^3 - 31x^2 + 3x + 10$ by $x^3 + 4x^2 + 6x - 5$.

5. $3x^{2y} + 5x^yz^c + 2z^{2c}$ by $3x^y + 2z^c$.

6. $10a^x + 11a^{x-1}y + 9a^{x-3}y^3$ by $2a + 3y$.

MISCELLANEOUS EXAMPLES FOR PRACTICE.

Divide the following:

1. $6\,(a + b)^nb^2c^{-2}$ by $3\,(a + b)^{-m}b^{-n}c^m$.

2. $14\,(x^2 + 2xy + y^2)^{a-x}b^{-n}c^2$ by $-7\,(x+y)^{a-x}bc^{-7}$.

3. $-72xy^{-3}\,(\sqrt{a-b} + \sqrt{x+b})^{-n}$ by $-8x^{-n}y^{-3}(\sqrt{a-b} + \sqrt{a+b})^3$.

4. $-24a^3\,(x^2-y^2)^3\,(a+b)^m$ by $4a^x\,(x-y)^3(a+b)^m$.

5. $56b^{-3}m^3\,(x+y)^4$ by $8b^m m\,(x+y)^m\,(x-y)^3$.

6. $-15k^7l^{-m}\,(a^2-2ab+b^2)^{-n}(x-a)^n$ by $5k^al^{-3}(a-b)^{-n}(x-a)^x$.

7. $32a^2b^{-5}(x-y)^{\frac{1}{2}}\,(c+a)^{\frac{1}{3}}$ by $-4a^{-7}b^n(x-y)^{\frac{1}{7}}\cdot(c+a)^{\frac{1}{3}}$.

8. $-105a^3k^2\,(x+y)^{\frac{1}{m}}\,(a^2-b^2)^{\frac{1}{n}}$ by $-7a^3k^{-5}(x+y)\,(a+b)^{\frac{1}{n}}$.

9. 2^6 by 4.

10. $\left(\frac{3}{5}\right)^7$ by $\left(\frac{6}{25}\right)^6$.

11. $4a^4 + 3a^3b + 8a^2b^2 - 3ab^3 + 18b^4$ by $a^2 + 2ab + 3b^2$.

12. $4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4$ by $x^3 + 2x^2 + 3x + 4$.

13. $x^5 - 7x^4 + 3x^3 + 18x^2 + 37x + 20$ by $x^2 - 5x - 4$.

14. $\frac{2}{5}x^3 + \frac{53}{10}x^2 + \frac{7}{2}x - \frac{25}{8}$ by $\frac{2}{5}x + 5$.

15. $a^2 - b^2$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

16. $a^{3x} + 3a^{2x}b^{y} + 3a^{x}b^{2y} + b^{3y}$ by $a^{x} + b^{y}$.

17. $-a^{-7}b + a^{-6}b^{3} + 14a^{-5}b^{5} + a^{-4}b^{7} - 15a^{-3}b^{9}$ by $-a^{-4}b^{-1} + 3a^{-3}b + 5a^{-2}b^{3}$.

18. $x^{2a-b}y^{c+d} - 7x^{2a}y^{c+2d} + 13x^{2a+b}y^{c+3d} - 15x^{2a+2b}y^{c+4d}$ by $x^{a}y^{d} - 5x^{a+b}y^{2d}$.

90. Special Cases in Division.

Divide $a^{n} - b^{n}$ by $a - b$ when n is a positive integer and a and b have any value.

Operation.

$$
\begin{array}{l}
a^{n} - b^{n} \,)\, \underline{a - b} \\
\qquad\qquad a^{n-1} + a^{n-2}b + a^{n-3}b^{2} \ldots\ldots\ldots a^{n-n}b^{n-1} \\
\underline{a^{n} - a^{n-1}b} \\
\qquad a^{n-1}b - b^{n} \quad \text{First remainder.} \\
\qquad \underline{a^{n-1}b - a^{n-2}b^{2}} \\
\qquad\qquad a^{n-2}b^{2} - b^{n} \quad \text{Second remainder.} \\
\qquad\qquad \underline{a^{n-2}b^{2} - a^{n-3}b^{3}} \\
\qquad\qquad\qquad a^{n-3}b^{3} - b^{n} \quad \text{Third remainder.} \\
\qquad\qquad\qquad \underline{\ldots\ldots\ldots\ldots} \\
\qquad\qquad\qquad a^{n-n}b^{n} - b^{n} \quad \text{nth remainder.}
\end{array}
$$

Since $a^{n-n} = a^{0} = 1$, the nth remainder reduces to $b^{n} - b^{n} = 0$.

Hence $a^{n} - b^{n}$ is exactly divisible by $a - b$.

The fact of this exact division, expressed for certain values of a and b, gives us the following special principles:

When a and b represent positive quantities.

Principle I. The difference of the same powers of any two quantities is divisible by the difference of the quantities.

When a represents a positive and b a negative quantity, then

$$\frac{a^n - b^n}{a - b} = \frac{a^n - (-b)^n}{a - (-b)}$$

$$= \frac{a^n - b^n}{a + b} \text{ when } n \text{ is an even number,}$$

$$= \frac{a^n + b^n}{a + b} \text{ when } n \text{ is an odd number,}$$

b, as well as a, being now considered a positive quantity. Hence,

Principle II. The difference of the same even powers of any two quantities is divisible by the sum of the quantities.

Principle III. The sum of the same odd powers of any two quantities is divisible by the sum of the quantities.

If $-b$ is substituted for b in the quotient of the general example the sign of each term containing an odd power of this negative quantity will become —. The results of the division in the cases covered by the three special principles can be expressed algebraically by the following formulas in which a and b represent positive quantities:

Formula 1.

When n equals any positive integer.

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 \ldots\ldots\ldots\ldots + b^{n-1}$$

Formula 2.

When n equals any even positive integer.

$$\frac{a^n - b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots\ldots\ldots - b^{n-1}$$

Formula 3.

When n equals any odd positive integer.

$$\frac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 \ldots\ldots\ldots\ldots + b^{n-1}$$

The second members of these equations translated into common language give us the following rule for writing out the quotient:

RULE.

1. *When the divisor is the difference of two quantities, connect the terms of the quotient by the plus sign; when the divisor is the sum of two quantities, write the terms alternately plus and minus.*

2. *In the first term of the quotient write the first term of the divisor with an exponent one less than its exponent in the dividend and in the following terms decrease its exponent continually by one.*

3. *In the second term of the quotient write the first power of the second term of the divisor and in the following terms continually increase its exponent by unity until in the last term its exponent equals its exponent in the dividend diminished by one.*

Remarks on Examples.

1. It must always be borne in mind that the terms of the dividend are equal powers of the corresponding terms of the divisor. Thus if the first term of the dividend is the fifth power of the first term of the divisor, then the second term of the dividend is the fifth power of the second term of the divisor.

2. To find the exponent of this power in any given example, it is necessary to determine by the laws of multiplication how many times each term of the divisor must be taken as a factor in order to produce the corresponding term of the dividend. This can readily be done by inspection in the examples given.

3. When the exact divisor of a given binomial is to be found, careful examination of the terms of the binomial will reveal the equal powers involved. The terms of the divisor must be quantities which taken as factors the number of times indicated by the exponent of the power will give the corresponding terms of the dividend. After these quantities have been determined by inspection they must be connected by the proper sign as made known by the special principles.

4. Attention is called to the number of terms in the quotient.

In explaining examples:

1. State the equal powers involved in the terms of the dividend.

2. State the special principle used.

3. Give the rule by which the quotient is determined.

4. Point out the examples which do not come under any of the special principles.

Illustrative Examples.

1. Divide $a^6 - b^6$ by $a - b$.

$$\frac{a^6 - b^6}{a - b} = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5.$$

2. Divide $x^{10} + y^5$ by $x^2 + y$.

$$\frac{x^{10} + y^5}{x^2 + y} = x^8 - x^6y + x^4y^2 - x^2y^3 + y^4.$$

3. Divide $625a^4x^4 - 16b^4$ by $5ax + 2b$.

$$\frac{625a^4x^4 - 16b^4}{5ax + 2b} = 5^3a^3x^3 - 5^2a^2x^2 2b + 5ax 2^2b^2 - 2^3b^3.$$
$$= 125a^3x^3 - 50a^2bx^2 + 20ab^2x - 8b^3.$$

EXAMPLES.

Write by inspection the quotients in the following examples:

1. Divide $x^3 - y^3$ by $x - y$.

2. Divide $x^4 - y^4$ by $x + y$.

3. Divide $m^7 + n^7$ by $m + n$.

4. Divide $m^4 + n^4$ by $m + n$.

5. Divide $a^5 - b^5$ by $a - b$.

6. Divide $a^6 + b^3$ by $a^2 + b$.

7. Divide $m^4 - n^8$ by $m - n^2$.

8. Divide $x^6 + y^6$ by $x^2 + y^2$.

9. Divide $x^{15} + y^{10}$ by $x^3 + y^2$.

10. Divide $y^{10} - z^{10}$ by $y^5 + z^5$.

11. Divide $8a^3 - b^3$ by $2a - b$.

12. Divide $x^3 + 27y^3$ by $x + 3y$.

13. Divide $16y^4 - 81z^4$ by $2y - 3z$.

14. Divide $64a^6 + y^6$ by $2a + y$.

15. Divide $32a^{10}b^5 + 243c^{15}$ by $2a^2b + 3c^3$.

16. Divide $\frac{1}{8}a^3 - 8y^3$ by $\frac{1}{2}a - 2y$.

17. Divide $16x^{4m} - y^{4n}$ by $x^m + y^n$.

18. Divide $64x^{24} - 729a^{12m}$ by $2x^4 - 3a^{2m}$.

Find by inspection the exact divisors of the following:

1. $b^3 + y^3$.

2. $a^{12} - b^3$.

3. $x^9 + y^6$.

4. $32x^5 - \frac{1}{32}y^5$

5. $81a^{10}y^5 - 16x^{15}z^{20}$. no exact divisor.

6. $243a^{5m} + 32b^{5n}$.

7. $.0016a^4 - 16b^4$.

CHAPTER V.

FACTORING.

91. Factoring is the process of resolving a quantity into its factors.

A factor of a quantity will exactly divide the quantity. It is therefore a **divisor** of the quantity, and conversely a divisor of a quantity is a factor of that quantity.

The **factors** of a quantity multiplied together will produce the quantity. The correctness of the work of factoring can thus be easily tested.

The factors may be **prime** or **composite.**

Composite factors of a quantity may be resolved into other factors, and thus the quantity itself be finally separated into its prime factors.

The quantities to be factored may be monomials or polynomials.

A monomial may be factored by inspection. Thus the factors of $4a^2bc$ are 2, 2, a, a, b and c.

Polynomials are factored by inspection or by rules growing out of the processes of multiplication and division.

The following cases are considered in this chapter:

92. Case I. *When one of the factors is a monomial.*

Since dividing all the terms of a quantity divides the quantity, it follows that a monomial, common to all the terms of the polynomial, is a *divisor* and consequently a *factor* of that polynomial. The other factor can be obtained by division. The polynomial may be placed equal to the indicated product of the two factors, the polynomial factor being inclosed in a parenthesis.

Thus,

$$3abc+6a^3b^2=3ab\ (c+2a^2b).$$

Again,

$$2x^2y^2+6axy-4abx^3y=2xy\ (xy+3a-2abx^2).$$

Remarks on Examples.

1. In all examples, the monomial factor should be the one first taken out.

2. The monomial factor removed should be the one of highest degree.

3. Striking out a monomial factor from part of the terms of a polynomial is not sufficient; it does not divide the polynomial.

4. The factors obtained should be plainly designated. It is well at the close of the process to place the given polynomial equal to the indicated product of the factors found.

EXAMPLES.

Factor the following:

1. $2ab^2+3bc$.
2. $5a^2b+10ab^2+15a^2b^2$.

3. $\frac{1}{2}x^2y^3z - \frac{1}{4}xy^2z^3 + \frac{3}{4}xy^3z^2$.

4. $2am^2\sqrt{x-a} + 4a^3m^3\sqrt{x+y}$.

5. $2ab^3 + 10a^2b^2(2x-3a)$.

6. $3m^2n(x-a) + 6mn^2(y-z)$.

7. $(x-3)y + (x-3)z$.

8. $(m+2)(m-2) - (m+2)(m+3)$.

93. Case II. *When the quantity to be factored consists of uncollected terms and is the product of two polynomials.*

In multiplying together two polynomials each term of one polynomial is in succession multiplied into the other polynomial. If the terms of the product are uncollected they can, in whatever order given, be arranged in sets, each resolvable into a monomial times this latter polynomial. The algebraic sum of these monomial coefficients into the polynomial will give the original quantity, factored as required.

Notice the following multiplication:

$$\begin{array}{l} \quad a+b \\ \quad \underline{c+d} \\ ca+cb \\ \qquad \underline{da+db} \\ ca+cb+da+db \end{array}$$

In the above c and d are multiplied in succession into $a+b$, the first polynomial. The same result is obtained if a and b are multiplied in succession into $c+d$, the second polynomial. Since there are two methods of multiplying the factors to form the product, there are evidently two methods of resolving the product into its factors, hence two solutions to all examples of this class.

Thus,

First solution, $ca+cb+da+db=c(a+b)+d(a+b)$ $=(c+d)(a+b)$.

Second solution,

$$\begin{aligned} ca+cb+da+db &= ca+da+cb+db \\ &= a(c+d)+b(c+d) \\ &= (a+b)(c+d) \end{aligned}$$

EXAMPLES.

Find the factors of the following, giving two solutions to each example:

1. $mx-nx+my-ny$.
2. $ax^2y^2-az+cx^2y^2-cz$.
3. $3x^4+3x^3-x^2-x$.
4. $ax+az-by+ay-bz-bx$.
5. $2x^2+xz+3xy+az+2ax+3ay$.
6. $8ax-12ay-10bx+15by$.
7. $(x+y)^2-ax-ay$.
8. $ax+a(z-c)-bx-b(z-c)$.

94. Case III. *When the quantity to be factored is the square of a binomial.*

It has been found that the square of a binomial is equal to the algebraic sum, of the square of the first term of the binomial, twice the product of the first term into the second term, and the square of the second term. The signs of the terms which are perfect squares must necessarily be plus, while the sign of the remaining term must be the same as that between the terms of the binomial. It follows then that if any algebraic expression can be put into the form of a trinomial, in which two of the terms are perfect squares preceded by the plus sign, and the remaining term is twice the product of their square roots preceded by either sign, the two equal factors of the expression will be the square roots of the terms which are perfect squares, connected by the sign preceding the remaining term of the trinomial.

It should be noticed that the terms of the trinomial may themselves be either monomials or polynomials.

EXAMPLES.

Factor the following:

1. $4x^2+12xy+9y^2$.
2. $16a^2+25b^2-40ab$.
3. $(a+b)^2+2(a+b)(x+y)+(x+y)^2$.
4. $y^{2m}+2y^m+1$.
5. $81z^{2n}+9y^2-54yz^n$.
6. $9(x+y)^2-3(x+y)+\frac{1}{4}$.

7. $\frac{1}{25}+25(x-a)^2+2(x-a)$.

8. $\frac{2}{3}(m+n)+(m+n)^2+\frac{1}{9}$.

95. Case IV. *When the quantity to be factored is the difference of two squares.*

(Since the product of the sum and difference of any two quantities is equal to the difference of their squares, it follows that, any quantity which is the difference of two squares can be separated into two factors, one of which is the sum and the other the difference of the square roots of these squares.)

EXAMPLES.

Factor the following:

1. a^2-9b^2.
2. $25x^2-36y^2z^2$
3. $\frac{1}{4}a^2-\frac{1}{9}y^2$.
4. $a^{2m}-b^2$.
5. $(a+b)^2-c^2$.
6. $x^2y^2-(a-b)^2$.
7. $m^2+2mn+n^2-a^2+2ab-b^2$.
8. $x^2-(z-c)^2+y^2-2xy$.
9. $9a^2+12ab-c^2+4b^2$.
10. $(x^2+y^2-z^2)^2-4x^2y^2$.
11. $(x^2-a^2+y^2-b^2)^2-8abxy-4x^2y^2-4a^2b^2$.

Expressions can often be put into the form covered by this case, by the addition and subtraction of the same quantity.

Thus,

$$a^2+4a-5=a^2+4a+4-9=(a+2)^2-(3)^2$$
$$=(a+2+3)(a+2-3)=(a+5)(a-1)$$

12. x^2+8x-9.

13. $4x^2+12x-7$.

14. $9a^2-12a+3$.

96. Case V. *When the quantity to be factored is the product of two binomials whose first terms are the same.*

The product of two binomials having a common first term is equal to the square of the first term, plus the algebraic sum of the second terms into the first, plus the product of the second terms.

Thus,

$$(x+2)(x+3)=x^2+(2+3)x+6$$
$$=x^2+5x+6.$$

It follows that the factors of such a quantity are two binomials each having as its first term the square root of the first term of the quantity, and as second terms respectively, the two factors of the third term of the quantity which algebraically added equal the coefficient of the second term of the quantity.

Thus, required the factors of $m^2+6m-16$.

The first term of each factor is the square root of m^2 or m. The second terms respectively are $+8$ and

-2, factors of the third term, -16, which added give $+6$, the coefficient of the second term of the quantity.

Hence $m^2+6m-16=(m+8)(m-2)$.

EXAMPLES.

Factor the following, explaining fully how each term of each binomial is obtained:

1. $x^2-11x+30$.
2. $x^2-4x-21$.
3. $x^2+4x-32$.
4. $x^2+7bx+12b^2$.
5. $y^2-33ay+200a^2$. 8 & 25
6. $a^2-25abc-150b^2c^2$. 5 & 30
7. $(x+y)^2+23(x+y)-78$.
8. $(a-b)^2-19(a-b)-120$.

97. Case VI. *When the expression to be factored is the difference of the same odd powers of two quantities.*

Such an expression has been shown to be divisible by the difference of the quantities. One factor is thus made known; the other is the quotient obtained by division.

Remarks on Examples.

1. The utmost care should be exercised in deter-

mining the quantities whose equal powers are involved in the terms of the binomial to be factored.

2. The student should write the second factor by the special rule given in division.

3. The difference of the same *even* powers of quantities should not be included under this case. Such examples can be solved to much greater advantage under case IV.

EXAMPLES.

Factor the following:

1. $x^3 - y^3$.
2. $32x^5 - y^5$.
3. $a^{10} - 243y^5$.
4. $27a^3c^6 - 8b^3$.
5. $(x+y)^3 - 8z^{12}$.
6. $32(a-b)^5 - (a+b)^5$.
7. $64(x+y)^6 - 125(x-y)^3$.

98. Case VII. *When the expression to be factored is the sum of the same odd powers of any two quantities.*

By the third special principle in division such expression is known to be exactly divisible by the sum of the quantities. The quotient or other factor can be written out under the rule.

EXAMPLES.

Factor the following:

1. x^3+a^3.

2. $8a^3+b^3$.

3. $8x^6+27b^3$.

4. $32x^{10}+243y^{15}$.

5. $64a^9+125(a+b)^3$.

6. $128(x-y)^7+(x+y)^7$.

99. Remarks on Miscellaneous Examples.

1. Success in solving examples is largely dependent on skill in grouping terms and introducing parentheses, so as to clearly bring the given expression under some one of the foregoing cases.

2. The solution of a single example will often involve the repeated use of the same principle or the employment in succession of different principles.

3. The principles used and the order of their use can only be determined by careful inspection. It is convenient, in general, after removing the monomial factor, to note whether the remaining expression can be put into the form of the difference of two squares.

4. The factors multiplied together must, of course, equal the original quantity. The simple calling to mind of this test will often prevent the student from omitting or duplicating factors in the final expression of the quantity factored.

MISCELLANEOUS EXAMPLES.

Resolve the following into prime factors:

1. $2x^2+4xy+2y^2$.
2. $(x-y)^2-6(x-y)+9$.
3. a^6-b^6.
4. $(2x+3)^2-(x-5)^2$.
5. $3ax^2y-15axy-72ay$.
6. $2xy-5ay+4bx-10ab$.
7. $4a^2+4a+1-x^2$.
8. $y^2-9x^2+30ax-25a^2$.
9. $2ab^2x^2+28abx-102a$.
10. $4x^2-25m^2-4n^2+9y^2+20mn-12xy$.
11. x^7+32x^2.
12. $x^4-(5x-6)^2$.
13. $(x^2-5x)^2-2(x^2-5x)-24$.
14. $16x^2+24x-16$.
15. $36a^2c^2-81b^2c^2-16a^2d^2+36b^2d^2$.
16. $(a^2+9)^2-36a^2$.
17. $9x^2-12xy+4y^2-3xz+2yz$.
18. $(x^2+y^2-9)^2-4x^2y^2$.
19. $(3a-2b)^2+3(9a^2-4b^2)$.

20. $a^8b^{16}-17a^4b^8+16$:

21. $(x^2-3x+14)^2-144x^2$.

22. $(4x^2-9)^2-(2x-3)^2(4x+5)$.

23. $(x^{2a}+y^{2b}-z^{2c})^2-4x^{2a}y^{2b}$.

24. $x^{12}y^2+y^{14}$.

25. $x^2-y^2-z^2+v^2-2(xv-yz)$.

26. $x^{12}-x^8y^4-x^4y^8+y^{12}$.

27. $a^8+56a^5-512a^2$.

28. $a^3-b^3-3b^2(a-b)$.

29. $9y^2-x^2+x+3y$.

30. $8a^3-27b^3-4a^2+9b^2$.

31. $16(a^2-a)^2-8(a^2-a)-3$.

CHAPTER VI.

HIGHEST COMMON FACTOR.

100. A common factor of two or more quantities is a factor which is found in each of them.

Thus, a is a common factor of the three quantities ab, abc, and acd, since it is a factor found in each of them.

If the common factor is composite, each of its prime factors is a common factor of the quantities.

Thus, since the composite factor 6 is common to the quantities $6ab = 2 \times 3ab$ and $6dc = 2 \times 3dc$, it follows that its prime factors 2 and 3 must be common factors of those quantities.

The product of *any number* of prime factors, common to two or more quantities, is a *common factor* of those quantities.

Thus, since the quantities $2ac^2$, $2abc$ and $2acx$ contain the common prime factors 2 and a, their product, $2a$, must appear as a common factor in those quantities.

The product of *all* the prime factors common to two or more quantities is the *highest common factor* of the quantities.

Thus, in the example given above $2ac$ is the highest common factor. There is no higher common factor, for if there were, its prime factors would be found in each of the quantities; but all the common prime factors of the quantities have been taken and are contained in $2ac$.

101. The Highest Common Factor (H. C. F.) of two or more quantities is, then, the common factor of highest degree which has, also, the numerical coefficient of greatest absolute value.

The sign of the highest common factor can be changed and it still contain all the common prime factors. The highest common factor has then two forms, one positive and the other negative.

102. Quantities having no common factor other than unity are said to be **prime** to each other.

Thus, $2ab$ and $3cd$ are prime to each other.

Two cases arise in determining the highest common factor of algebraic quantities.

103. Case I. *When the quantities are monomials, or polynomials readily resolvable by inspection into their prime factors.*

In this case we have the following

RULE.

Resolve the quantities into their prime factors and find the continued product of those factors that are common to all the quantities. The result will be the highest common factor required.

EXAMPLES.

1. Find the H. C. F. of $6a^2b^3x$ and $9ab^4x^2$.

Resolving the quantities into their prime factors, we have

$$6a^2b^3x = 2 \times 3 \times a \times a \times b \times b \times b \times x$$
$$9ab^4x^2 = 3 \times 3 \times a \times b \times b \times b \times b \times x \times x.$$

The common factors are 3, a, b, b, b, and x. Hence the H. C. F. is $3ab^3x$.

2. Find the H. C. F. of $3an^2+3bn^2$ and $6an+6bn$.

Resolving the quantities into their prime factors, we have

$$3an^2+3bn^2 = 3 \times n \times n \times (a+b).$$
$$6an+6bn = 3 \times 2 \times n \times (a+b).$$

Hence $3n\ (a+b)$ is the H. C. F.

3. Find the H. C. F. of $a^2-3ab+2b^2$, $a^2-2ab+b^2$ and a^2-b^2.

Factoring we have

$$a^2-3ab+2b^2 = (a-b)\ (a-2b).$$
$$a^2-2ab+b^2 = (a-b)\ (a-b).$$
$$a^2-b^2 = (a-b)\ (a+b).$$

Hence $a-b$ is the H. C. F.

4. Find the H. C. F. of $18ab^2c^2d$ and $36a^2bcd^2$.

5. Find the H. C. F. of $8x^2y^3z^4$, $12x^3y^2z^3$ and $20x^5yz^2$.

6. Find the H. C. F. of a^2-b^2 and a^3-b^3.

7. Find the H. C. F. of $x^2+6x-27$ and $x^2+11x+18$.

8. Find the H. C. F. of $x^2-3x-18$, x^3-9x and x^3+27.

9. Find the H. C. F. of $a^3-2a^2b+ab^2$, $a^4-a^2b^2$ and $2a^3+2a^2b-4ab^2$.

10. Find the H. C. F. of $ab-3b-5a+15$, $a^4-6a^3+9a^2$ and a^5-243.

11. Find the H. C. F. of $(x^2+y^2-z^2)^2-4x^2y^2$, $2ax^2+2ay^2-2az^2+4axy$, and $y^2+2yz+z^2-x^2$.

12. Find the H. C. F. of $xy-2ay-3cx+6ac$, $x^2+3ax-10a^2$ and $x^2y+cx^2-4a^2y-4a^2c$.

104. Case II. *When the quantities are polynomials whose factors cannot be readily discovered.*

The rule for finding the highest common factor in this case is based upn the following preliminary principles:

Principle I. A factor of a quantity is also a factor of that quantity multiplied by another. (Prin. I. Division.)

Thus,

2 is a factor of 6 $\therefore$ a factor of 6×5.

a is a factor of ab $\therefore$ a factor of abc.

Principle II. A common factor of two quantities is also a factor of their sum and of their difference. (Prin. II. and III. Division.)

Thus,

3 is a factor of 9 and 6 $\therefore$ a factor of $9+6$ and $9-6$.

a is a factor of ab and ac $\therefore$ a factor of $ab+ac$ and $ab-ac$.

By the aid of these principles the following proposition can be proven:

Proposition. *The H. C. F. of two quantities is the same as the H. C. F. of the quantity of lowest degree and the remainder after division.*

Expressing the division algebraically thus,

$$\begin{array}{c} b\,)\,a\,(\,q \\ \underline{\;qb\;} \\ r \end{array}$$

and representing the H. C. F. of b and r by h we are required to prove:

1. That h is a common factor of a and b.

2. That h is the *highest* common factor of a and b.

Proof of 1. h divides b and r
$\therefore$ Prin. I. h divides bq and r.

$\therefore$ Prin. II. h divides $bq+r$ or a
$\therefore$ h is a common factor of a and b.

Proof of 2. If h is not the H. C. F. of a and b, then hl (h multiplied by some additional common factor as l) must be such H. C. F. If this be true it follows that:

hl divides a and b
$\therefore$ Prin. I. hl divides a and bq.

$\therefore$ Prin. II. hl divides $a - bq$ or r

$\therefore$ hl is a common factor of b and r which is absurd, since h is the H. C. F. of these quantities. The supposition at the beginning of proof of 2, that h is *not* the H. C. F. of a and b is therefore false and hence it is proven that h, *the H. C. F. of b and r, is the H. C. F. of a and b.*

Granting this proposition and continuing the division with b and r as the new dividend and divisor we have,

$$\begin{array}{l}
r)\,b\,(q' \\
\quad\underline{q'r} \\
\qquad r')\,r\,(q'' \\
\qquad\quad \underline{q''r'} \\
\qquad\qquad r'')\,r'\,(q''' \\
\qquad\qquad\quad \underline{q'''r''} \\
\qquad\qquad\qquad\quad 0
\end{array}$$

in which by the proposition:

The H. C. F. of b and r is the H. C. F. of r and r', and the H. C. F. of r and r' is the H. C. F. of r' and r'', but r'' (representing the last divisor) is exactly divisible into r' and is therefore the H. C. F. of the last pair of quantities r'' and r'. Since it is also the H. C. F. of each preceding dividend and divisor in the successive divisions, it is the H. C. F. of the two original quantities a and b.

Before employing this principle to find the H. C. F.

of any two polynomials, the following remarks upon the proposition should receive attention:

1. The terms of the polynomials should, for convenience, be arranged according to the descending powers of some common letter.

2. Since the highest common factor of two quantities is the product of all their common factors, it must contain any monomial factor common to the two. Such monomial factor should, however, be removed and afterward multiplied into the H. C. F. of the resulting quotient.

3. Any monomial factor common to the *terms* of one quantity but not common to the two quantities should be discarded. This will evidently not affect the highest common factor.

4. If, at any time, the first term of the properly arranged dividend is not divisible by the first term of the divisor, the dividend should be multiplied by such monomial factor as will render its first term divisible by the first term of the divisor. If all monomial factors have been previously removed from both polynomials, the introduction of this factor into the dividend will not change the common factors, and cannot, therefore, affect the highest common factor.

The principles set forth in the preceding proposition and remarks are embodied in the following general

RULE.

1. *Arrange the terms of the two polynomials according to the descending powers of some common letter; remove the*

monomial factors of each polynomial, reserving such as are common to the two as factors of the highest common factor, and discarding those that are found in only one polynomial.

2. *If necessary, multiply the polynomial of highest degree by such monomial factor as will render its first term divisible by the first term of the second polynomial, and dividing the first polynomial by the second, continue the division until the remainder is of lower degree than the divisor.*

3. *Divide this remainder into the first divisor proceeding as in the first division. Continue to divide in succession the last remainder into the preceding divisor until the division is complete. The product of the remainder which serves as the last divisor, into the common monomial factor, if any, will be the H. C. F. required.*

EXAMPLES.

1. Find the H. C. F. of $x^4+4x^3+5x^2+2x$ and x^3+5x^2+4x.

Removing the factor x from each, and reserving it as a part of the H. C. F., we proceed thus:

$$\begin{array}{l|l}
x^3+4x^2+5x+2 & \underline{x^2+5x+4} \\
 & x-1 \\
\underline{x^3+5x^2+4x} & \\
\quad -x^2+x+2 & \\
\quad \underline{-x^2-5x-4} & \\
\qquad\qquad 6x+6 &
\end{array}$$

Removing the factor 6 in this remainder and discarding it, we have $x+1$ for the next divisor.

$$\begin{array}{l|l}
x^2+5x+4 & \underline{x+1} \\
\underline{x^2+x} & x+4 \\
\qquad 4x+4 & \\
\qquad \underline{4x+4} &
\end{array}$$

Here the division is exact; hence $x+1$ is the H. C.

F. of x^3+4x^2+5x+2 and x^2+5x+4; and combined with the reserved factor x gives $x(x+1)$ as the H. C. F. required.

2. Find the H. C. F. of $16x^3-44x^2+10x$ and $18x^3-87x^2+129x-60$.

Removing from the first polynomial the monomial factor $2x$, we have $8x^2-22x+5$; removing from the second the factor 3, we have $6x^3-29x^2+43x-20$.

Introducing the factor 4 into the second polynomial to make its first term divisible by the first term of the other, we proceed thus:

$$\begin{array}{rrrrl}
24x^3 & -116x^2 & +172x & -80 & (\underline{8x^2-22x+5} \\
\underline{24x^3} & \underline{-66x^2} & \underline{+15x} & & \;\;3x-25 \\
 & -50x^2 & +157x & -80 & \\
\text{Introducing } 4 & & & 4 & \\
 & -200x^2 & +628x & -320 & \\
 & \underline{-200x^2} & \underline{+550x} & \underline{-125} & \\
 & & 78x & -195 &
\end{array}$$

Removing the factor 39 in this remainder, we have $2x-5$ for the next divisor.

$$\begin{array}{rrl}
8x^2 & -22x+5 & (\underline{2x-5} \text{ H. C. F.} \\
\underline{8x^2} & \underline{-20x} & \;\;4x-1 \\
 & -2x+5 & \\
 & \underline{-2x+5} &
\end{array}$$

3. Find the H. C. F. of $72a^6b^2+63a^5b^2-144a^4b^2+9a^3b^2$ and $18a^5b^2+9a^4b^2-27a^3b^2$.

4. Find the H. C. F. of $x^4+x^3+2x^2-x+15$ and $3x^4+9x^3-12x^2+27x+27$.

5. Find the H. C. F. of $6ax^4-2ax^3-2ax^2-2ax-8a$ and $6ax^4-14ax^3+2ax^2+2ax+8a$.

6. Find the H. C. F. of $10a^3 - 11a^2 - 55a - 6$ and $4a^2 - 5a - 21$.

7. Find the H. C. F. of $6x^3 - 11x^2 - 5x + 12$ and $4x^3 - 18x^2 + 34x - 24$.

8. Find the H. C. F. of $5x^4 - 30x^3 - 34x^2 + 198x + 36$ and $5x^3 - 20x^2 - 64x + 24$.

9. Find the H. C. F. of $18x^5 - 132x^4 + 126x^3 - 36x^2 + 80x$ and $18x^5 - 150x^4 + 150x^3 + 6x^2 + 24x$.

10. Find the H. C. F. of $6x^4 + 19ax^3 + 3a^2x^2 + 8a^3x + 24a^4$ and $6x^3 + 10ax^2 - 17a^2x + 21a^3$.

To find the highest common factor of three or more polynomials, find the highest common factor of two of them, then of this result and a third and so on. The final divisor will be the highest common factor required.

EXAMPLES.

1. Find the H. C. F. of $x^3 + x^2 + x - 3$, $x^3 - x - 6$ and $x^3 - x^2 - 3x - 9$.

2. Find the H. C. F. of $2x^3 + 5x^2 + x - 2$, $2x^3 - 5x^2 - 4x + 3$ and $2x^3 + x^2 - 16x - 15$.

3. Find the H. C. F. of $2x^4 + 5x^3 - 7x^2 + 10x - 4$, $2x^4 - 7x^3 - x^2 - 6x + 4$ and $4x^4 - 8x^3 + 7x^2 + 6x - 4$.

4. Find the H. C. F. of $3a^4 - 7a^3b + 5a^2b^2 - ab^3$, $3a^4 + 5a^3b + a^2b^2 - ab^3$ and $a^2b + 3ab^2 - 3a^3 - b^3$.

5. Find the H. C. F. of $6x^4 + x^3 - x$, $4x^4 - 6x^3 - 4x^2 + 3x$ and $2x^4 + x^3 + x^2 - x$.

CHAPTER VII.

LOWEST COMMON MULTIPLE.

105. A Multiple of a quantity is any expression of which the quantity is an exact divisor. It is the *quantity* multiplied by *any factor*.

Thus, 10 is a multiple of 2; $4ab$ is a multiple of ab.

106. A Common Multiple of two or more quantities is any expression of which each of the quantities is an exact divisor.

Thus, 6 is a common multiple of 3 and 2; $6xy$ is a common multiple of 6, x and y.

107. The Lowest Common Multiple (L. C. M.) of two or more quantities is the common multiple of lowest degree, having the numerical coefficient of least absolute value.

Thus, $12ab$ is the lowest common multiple of $4a$ and $6b$. It is a lower common multiple than $24ab$, for the reason that though of the same degree its numerical coefficient is of less absolute value.

It is obvious that changing the sign of the L. C. M. does not affect its degree nor the absolute value of its numerical coefficient. The L. C. M. has therefore two forms one of which is the negative of the other and either of which can be used as found convenient.

Thus, $12ab$ or $-12ab$ may be taken as the L. C. M. of $4a$ and $6b$.

108. The following principles are the evident outgrowth of the definitions and the laws of division:

Principle I. The multiple of a quantity must contain all the prime factors of that quantity.

Principle II. The common multiple of two quantities must contain the prime factors of each quantity. It may contain other factors.

Principle III. The *lowest* common multiple of two quantities must contain the prime factors of each quantity and no other factors.

These principles enable us to determine the L. C. M. of any two algebraic quantities. Two cases arise:

109. Case I. *When the quantities can be readily factored.*

In the product of any two quantities the common factors of those quantities must appear *twice* or in *two sets.* It is clear from principle III that such common factors occur but once in the L. C. M., one set having been rejected.

Hence the following

RULE.

Resolve the quantities into their prime factors. Reject one set of the factors common to the two quantities. The product of the remaining factors will be the L. C. M. required.

EXAMPLES.

1. Find the L. C. M. of $6x^2y$ and $9xy^2$.

Resolving the quantities into their prime factors, we have

$$6x^2y = 2 \times 3 \times x \times x \times y$$
$$9xy^2 = 3 \times 3 \times x \times y \times y$$

Rejecting one set of the factors (3, x and y) common to the quantities and taking the product of the remaining factors, we have for the L. C. M.

$$2 \times 3 \times 3 \times x \times x \times y \times y = 18x^2y^2.$$

2. Find the L. C. M. of $2(a-b)$ and $6(a^2-b^2)$.

3. Find the L. C. M. of $a^2-2ab+b^2$ and $a^2+ab-2b^2$.

4. Find the L. C. M. of x^2-y^2 and $cx+cy+dx+dy$.

5. Find the L. C. M. of m^3-n^3 and $m^2+2mn-3n^2$.

When the L. C. M. of more than two quantities is required, reject one set of the factors common to any two of the quantities, then a set common to that result and a third quantity and so on. The remaining factors multiplied together will give the L. C. M.

6. Find the L. C. M. of $a^2-2ab+b^2$, a^2-b^2 and $a^2+2ab-3b^2$.

Resolving into factors, we have

$$\begin{aligned} a^2-2ab+b^2 &= (a-b)(a-b) \\ a^2-b^2 &= (a-b)(a+b) \\ a^2+2ab-3b^2 &= (a-b)(a+3b) \end{aligned}$$

Rejecting the factor $a-b$ common to the first two quantities, we have $(a-b)(a-b)(a+b)$ as their

L. C. M. Proceeding as before with this result and the third quantity, we reject the common factor $a-b$ and have as the L. C. M. of the three quantities, $(a-b)(a-b)(a+b)(a+3b) = (a-b)^2(a+b)(a+3b)$.

7. Find the L. C. M. of $8a^4$, $10a^3b$ and $12a^2b^2$.

Ans. $120a^4b^2$.

8. Find the L. C. M. of $x+1$, x^2-1 and x^3-1.

9. Find the L. C. M. of $(a+b)^2$, a^2-b^2, $(a-b)^2$ and a^3-b^3.

10. Find the L. C. M. of $x^2-9x+18$, $x^2+3x-54$ and $x^2-3x-108$.

11. Find the L. C. M. of $x^2+7x+12$, x^2+x-12 and x^2+3x-4.

12. Find the L. C. M. of $(x+3y)^3$, $(x-3y)^3$ and x^2-9y^2.

110. Case II. *When the quantities cannot be readily factored.*

The rejection in Case I of a set of factors common to the two quantities is evidently equivalent to dividing their product by the H. C. F. In Case II the H. C. F. is determined by the method of continued division. When found, it is stricken out of the product, the same as in case I.

Hence the following

RULE.

Divide the product of the two quantities by their highest common factor and the result will be the lowest common multiple.

The division can in general be more readily effected by dividing one of the quantities (factors) than by dividing their product.

When the L. C. M. of more than two polynomials is required, find the L. C. M. of any two of them; then of that result and a third polynomial; and so on.

EXAMPLES.

Find the L. C. M. of $6x^3+5x^2+4x+15$ and $2x^2-5x-12$.

The highest common factor of the given quantities is $2x+3$.

Hence the L. C. M. is

$$\frac{(6x^3+5x^2+4x+15)(2x^2-5x-12)}{2x+3}$$
$$=(6x^3+5x^2+4x+15)(x-4).$$

2. Find the L. C. M. of a^4+a^2+1 and $a^4-a^3+2a^2+a+3$.

3. Find the L. C. M. of $2x^4+3x^3+3x-2$ and $3x^4+5x^3+x^2+5x-2$.

4. Find the L. C. M. of $x^5+x^3+x^2+x-2$ and $x^4-3x^3+6x^2-6x+4$.

5. Find the L. C. M. of $x^3-x^2-10x+6$, x^3-7x+6 and x^3+2x^2+9.

6. Find the L. C. M. of x^3+6x^2+3x-2, x^3+2x^2-5x-6 and x^3+3x^2-6x-8.

CHAPTER VIII.

FRACTIONS.

111. An algebraic fraction is an indicated division, expressed by writing the dividend above the divisor with a horizontal line between them.

Thus, $\frac{a}{b}$, in which a and b each represent any quantity, is a fraction.

The quantity above the line is the dividend and is called the *numerator*. The quantity below the line is the divisor and is called the *denominator*. Taken together, the numerator and denominator are called the *terms of the fraction*. The *value of the fraction* is the quotient arising from the indicated division.

112. A mixed quantity is one made up of an integer and a fraction; as $a+b+\frac{c}{d}$.

113. A complex fraction is one which has a fraction in either or both of its terms.

The following principles are important:

114. Principle I. *Multiplying the numerator or dividing the denominator of a fraction multiplies the fraction.*

Since, in division, the factors appearing in the quotient are those in the dividend less those in the divisor it is obvious that a factor introduced into the dividend (numerator) or stricken out of the divisor (denominator) will appear in the quotient; that is, the value of the fraction will be multiplied.

Thus, $\frac{4a}{2a} = 2,$

Introducing the factor a in the numerator $\frac{4a \times a}{2a} = 2a,$

or striking out the factor a from the denominator $\frac{4a}{2} = 2a.$

In either case the value of the fraction is multiplied by a.

115. Principle II. *Dividing the numerator or multiplying the denominator of a fraction divides the fraction.*

This evidently results from Principle I. since division is the converse of multiplication.

116. Principle III. *Multiplying or dividing both numerator and denominator of a fraction by the same quantity does not alter the value of the fraction.*

By Principles I. and II. this operation is equivalent to both multiplying and dividing the fraction by the same quantity which, of course, does not alter its value.

117. Signs of fractions.

The *real sign* of a fraction is the sign of the quotient of the expressed division. It is the resultant of the sign before the dividing line, called the *apparent sign*, the *sign of the numerator* and the *sign of the denominator*. These three signs we shall designate as the *signs of the fraction.*

The following are important principles governing their use:

Principle I. *Any two of the three signs of a fraction can be changed without altering the value of the fraction.*

Given $\frac{8}{2}=4$, $\frac{ab}{a}=b$,

changing the apparent sign and the sign of the numerator $-\frac{-8}{2}=4$, $-\frac{-ab}{a}=b$,

or changing the apparent sign and the sign of the denominator $-\frac{8}{-2}=4$, $-\frac{ab}{-a}=b$,

or changing the sign of the numerator and the sign of the denominator $\frac{-8}{-2}=4$, $\frac{-ab}{-a}=b$.

Hence the principle.

Principle II. *A change of one or of all three of the signs of a fraction will change the fraction from a positive to a negative quantity or vice versa.*

The proof of this is similar to the proof of Principle I.

Remarks on Application of Principles.

In changing the sign of the numerator or denominator, when composed of terms, change the sign of each term; if the term is a fraction, change its apparent sign only; if a parenthesis, change the sign before the parenthesis only; if composed of factors, change the sign of one or any odd number of factors.

2. Under Principle I, the signs of the numerator and denominator of a fraction can be changed at any time.

3. Principle I is often employed to great advantage in getting rid of the minus sign before the dividing line of a fraction. This is generally done by changing the apparent sign and the sign of the numerator.

The following examples illustrate the use of the principles of signs.

EXAMPLES.

Change the signs of numerator and denominator in the following:

1. $\dfrac{a+b-c-mn}{x-y+z}$ 2. $\dfrac{ax+by-c}{3a-4b-y}$

3. $\dfrac{3x-\dfrac{2a-4}{3a+2}-7}{4x-5y+7}$ 4. $\dfrac{4a-7b-(8x-3y+6)}{7y-8a+5}$

5. $\dfrac{6x-3y-\left(a-b-\dfrac{c}{d}\right)}{5a-7x}$

6. $\dfrac{(c-d)(e-f)(x-y)}{(a-b)(m-n)}$

Change the following into equivalent fractions with the plus sign before the dividing line:

7. $-\dfrac{3a-5b-7c}{2x-3y}$

8. $-\dfrac{4m-\dfrac{2x-a}{4}+7}{3a-5y+\dfrac{2a-3}{5}}$

9. $-\dfrac{6xy-7ab+(4a-7m)}{2bx-16mn}$ ←

10. $-\dfrac{5a-3km-\left(7a-\dfrac{2b+1}{c}\right)}{4a-7}$

11. $-\dfrac{(m+n)\ (x+y)\ (a-b)}{(c-d)\ (k-3)}$

Reduce the following so that the fractions in each example shall have a positive apparent sign and the same denominator:

12. $-\dfrac{(a-b)\ (a-c)}{(x-a)\ (x+a)}$ and $-\dfrac{(a-b)\ (a-c)}{(a-x)\ (a+x)}$.

13. $-\dfrac{(x-d)\ (x-e)}{(x-a)\ (x-c)}$ and $\dfrac{(x-b)\ (x-e)}{(x-a)\ (c-x)}$.

14. $\dfrac{(x+y)\ (x-z)\ (m-x)}{(m-y)\ (m^2-y^2)}$ and

$$\frac{(a+b)\ (a-b)\ (x-y)}{(y-m)\ \ (y^2-m^2)}.$$

15. $-\dfrac{(a-b)\ (c-d)}{(x^2-y^2)\ (x+y)}$, $-\dfrac{(a-c)\ (b-d)}{(y^2-x^2)\ (x+y)}$and

$$\frac{(a+d)\ (b+c)}{(x^2-y^2)\ (-x-y)}.$$

16. $-\dfrac{(x-d)\ (x-e)\ (x-f)}{(x-a)\ (x-b)\ (x-c)}$,

$$\frac{(x+d)(x-e)\ (x+f)}{(x-a)\ (b-x)\ (c-x)} \text{ and}$$

$$-\frac{(x+d)\ (x+e)\ (x+f)}{(a-x)\ (b-x)\ (c-x)}.$$

118. Algebraic operations connected with fractions will be considered under the following heads:

Reduction of Fractions.
Addition and Subtraction of Fractions.
Multiplication of Fractions.
Division of Fractions.
Reduction of Complex Fractions.

REDUCTION OF FRACTIONS.

119. To reduce a fraction is to change its *form* without altering its *value*.

The following are the more common cases met with:

120. Case I. *To reduce a fraction to its lowest terms.*

The numerator and denominator of a fraction in its lowest terms, are prime to each other. They cannot therefore contain any common factor.

Hence the

RULE.

To reduce a fraction to its lowest terms, separate its numerator and denominator into their prime factors and cancel those that are common.

Remarks.

1. To cancel a factor common to the terms of a fraction is equivalent to dividing its numerator and de-

nominator by the same quantity which, we have found, does not alter the value of the fraction.

2. Cancelling all factors common to the terms of a fraction is the same as dividing the numerator and denominator by their H. C. F.

3. When the terms of a fraction cannot be readily factored, this H. C. F. should be determined by division. The fraction can then be reduced to its lowest terms by dividing its numerator and denominator by this H. C. F.

EXAMPLES.

Reduce the following fractions to their lowest terms:

1. $\dfrac{27a^2b^3c^4}{45a^3bc^2}$

2. $\dfrac{a^3-x^3}{a^2-x^2}$

3. $\dfrac{a^2+7a+10}{a^2+5a+6}$

4. $\dfrac{a^3-1}{a^3+a^2+a}$

5. $\dfrac{ac+ad+bc+bd}{a^2-b^2}$

6. $\dfrac{(a+b)^2}{a^2-ab-2b^2}$

7. $\dfrac{a^2-(b+c+d)^2}{(a-b)^2-(c+d)^2}$

8. $\dfrac{(x-y)^2-(z-m)^2}{(x-y)^2-(x-y)(z-m)}$

9. $\dfrac{x^2-16}{x^2-x-20}$

10. $\dfrac{(x^2-9)(x^2-4x+3)}{(x^2-6x+9)\ (x^2+2x-3)}$

11. $\dfrac{(y^3+z^3)(y^2+yz+z^2)}{(y^3-z^3)(y^2-yz+z^2)}$

12. $\dfrac{a^2-(b-c)^2}{a^2-ab+ac}$

13. $\dfrac{(x^2-3x)^2-2(x^2-3x)-8}{(x^2-1)(x^2-6x+8)}$

14. $\dfrac{x^3-7x+6}{x^3-x^2-3x+2}$

15. $\dfrac{6x^3+7x^2+x+6}{2x^3+3x^2-4x-6}$

16. $\dfrac{2x^3-7x^2+11x-4}{3x^3-10x^2+15x-4}$

121. Case II. *To reduce a fraction to an entire or mixed quantity.*

We are required in this case to perform the division indicated by the fraction.

Hence the

RULE.

Divide the numerator by the denominator.

Remarks.

1. Such reduction is not advisable unless the numerator is of the same or a higher degree than the denominator.

2. The division should, in general, be continued until the remainder is of a lower degree than the divisor.

3. The remainder, if any, must be written above the divisor and this fraction annexed to the quotient.

4. The apparent sign of the fractional part of the result will at first be plus. It may be found convenient to change this sign and the sign of the numerator, under the principle of signs, in order to avoid a minus sign before the first term of the numerator.

Illustrative Example. Reduce $\frac{12x^2-5x-6}{3x^2+2}$ to a mixed quantity.

Process.

$$3x^2+2)12x^2-5x-6\,(4+\frac{-5x-14}{3x^2+2}=4-\frac{5x+14}{3x^2+2}$$
$$\underline{12x^2+8}$$
$$-5x-14$$

EXAMPLES.

Reduce the following to entire or mixed quantities:

1. $\frac{x^3-y^3}{x-y}$

2. $\frac{1-2x}{1-4x}$

3. $\frac{x^2-2x-3}{x^2+2x+3}$

4. $\frac{x^2-11x+25}{x-4}$

5. $\frac{x^4+4x^3y+6x^2y^2+4xy^3+y^4}{x^2+2xy+y^2}$

6. $\frac{x^{2a}-2x^ay^b-7y^{2b}}{x^a+2y^b}$

122. Case III. *To reduce a mixed quantity to a fraction.*

Viewing the mixed quantity as a result of the division under Case II. we have:

Integral part = quotient.

Denominator of fractional part = divisor.

Numerator of fractional part = remainder.

From the nature of division:

$$\frac{\text{Quotient} \times \text{divisor} + \text{remainder}}{\text{divisor}} = \frac{\text{dividend}}{\text{divisor}}.$$

Hence the

RULE.

Multiply the integral part by the denominator of the fraction, add the numerator to the product and write the result over the denominator.

Remarks.

1. Case III. may be considered as the converse of Case II. and the rule be thus derived.

2. If the apparent sign of the fractional part is minus, it can be treated in either of two ways: (a) The apparent sign and the sign of each term of the numerator can be changed under the principle of signs, or (b) Since the dividing line has the force of a parenthesis and the minus sign before it thus indicates that the entire quantity is to be subtracted, it follows that if the minus sign is retained, after reduction, the numerator which it affects must be inclosed

in a parenthesis. The parenthesis can then be removed when convenient.

Illustrative Example. Reduce $1-\frac{x-y}{x+y}$ to fractional form.

Process.

$$1-\frac{x-y}{x+y}=1+\frac{-x+y}{x+y}=\frac{x+y-x+y}{x+y}=\frac{2y}{x+y}$$

$$\text{or } 1-\frac{x-y}{x+y}=\frac{x+y-(x-y)}{x+y}=\frac{x+y-x+y}{x+y}$$

$$=\frac{2y}{x+y}$$

EXAMPLES.

Reduce the following to fractional form:

1. $a+\frac{1-a}{1+a}$

2. $2x-\frac{1-3x}{2x}$

3. $x+y+\frac{x^2}{x-y}$

4. $\frac{a^2-b^2}{a-b}-(a+b)$

5. $x-8-\frac{x^2-65}{x+8}$

6. $x+a-\frac{ax^2-a^2x}{x^2-a^2}$

7. $(x+1)^2+\frac{(x+1)^2}{x}$

8. $1-x+x^2-x^3+x^4-\frac{x^5-1}{1+x}$

123. Case IV. *To reduce fractions to equivalent fractions having the lowest common denominator.*

This is effected by a simple application of the principle that multiplying both numerator and denominator of a fraction by the same quantity does not alter the value of the fraction. The principle leads to the following

RULE.

Find the lowest common multiple of the denominators.

Divide this lowest common denominator by the denominator of the first fraction; multiply the quotient by the numerator of that fraction and write the result over the lowest common denominator.

Proceed in a similar way with each of the other fractions. The respective results will be the equivalent fractions required.

Remarks.

1. If the denominators of the given fractions have no common factor, the product of these denominators will be the common denominator and each numerator will be multiplied by the product of all the denominators except its own.

2. Fractions can be reduced to equivalent fractions having *any* common denominator. The lowest common denominator should, however, always be chosen.

3. If any of the given fractions can be reduced to lower terms, it should be done before finding the lowest common denominator. It is possible that the degree of the common denominator may be thus lessened

4. It will quite often be found convenient to leave the common denominator in its factored form. This is also true, to some extent, of the numerators of both the old and new fractions. In short, the student in all his algebraic work, before multiplying factors to-

gether, should be certain that the indicated is not preferable to the actual multiplication.

5. An entire quantity is regarded as having the denominator 1.

Illustrative Example. Reduce $\frac{2(1-x)}{1+x}$, $\frac{3a(1+x)}{(1-x^2)^2}$ and $\frac{(1+x)^2}{(1-x)^2}$ to equivalent fractions having the lowest common denominator.

Process.

$$\frac{2(1-x)}{1+x}=\frac{2(1-x)\;(1-x)^2}{(1+x)\,(1-x)^2}=\frac{2(1-x)^3}{(1+x)\,(1-x)^2}$$

$$\frac{3a\,(1+x)}{(1-x^2)^2}=\frac{3a\,(1+x)}{(1+x)^2\,(1-x)^2}=\frac{3a}{(1+x)\,(1-x)^2}$$

$$\frac{(1+x)^2}{(1-x)^2}=\frac{(1+x)\;(1+x)^2}{(1+x)\;(1-x)^2}=\frac{(1+x)^3}{(1+x)\,(1-x)^2}$$

EXAMPLES.

Reduce the following to equivalent fractions having the lowest common denominator:

1. $\frac{1}{2x^2y}$, $\frac{1}{6y^2z}$, $\frac{1}{2xz^2}$ and $\frac{1}{4x^2yz}$.

2. $\frac{3}{x-a}$, $\frac{4a}{(x-a)^2}$ and $\frac{5a^2}{(x-a)^3}$.

3. $\frac{a}{(a+b)b}$ and $\frac{b}{(a-b)a}$.

4. $\frac{x-2y}{x(y-x)}$, $\frac{2x+y}{y(x+y)}$ and $\frac{2x}{x^2-y^2}$.

5. $\frac{a}{a^2-x^2}, \frac{3b}{4a-4x}$ and $\frac{5x}{a+x}$.

6. $\frac{1+x}{1-x}, \frac{(1+x)^2}{1-x^2}$, and $\frac{1+x^3}{1-x^3}$.

7. $\frac{x-3}{x^2-3x+2}, \frac{x-2}{x^2-4x+3}, \frac{x-2}{x^2-5x+6}$.

8. $\frac{x+y}{(x-y)(z-x)}, \frac{x+z}{(y-x)(x-z)},$

$$\frac{y+z}{(x-y)(z-y)}.$$

ADDITION AND SUBTRACTION OF FRACTIONS.

124. It was proven in division that dividing all the terms of a quantity, divides the quantity. It is evident from this principle that when several fractions having a common denominator are to be *added*, this denominator may be written under each numerator or under their algebraic sum.

This clearly leads to the following

RULE.

Reduce the fractions to equivalent fractions having the lowest common denominator; find the algebraic sum of the numerators of these fractions and write the result over the common denominator.

In *subtraction* of fractions, the sign of the subtrahend must be changed; otherwise the process is the same as in addition.

Remarks.

1. The principle, that any two of the signs of a fraction can be changed without altering the value of the fraction, can be used to advantage in many of the examples in addition and subtraction. In general, this principle is helpful in most of the operations in which fractions are involved.

2. It is worthy of notice that the sign of an even number of factors of either numerator or denominator can be changed without altering the value of the fraction.

Thus, in adding

$\frac{ab}{(a-b)(c-d)}$ and $\frac{5c}{(b-a)(d-c)}$, a change in the signs of the two factors of either denominator will make the denominators the same. In applying this principle, it is better to arrange the letters, when possible, in alphabetical order, as,

$$\frac{5c}{(b-a)(d-c)} = \frac{5c}{(a-b)(c-d)}.$$

3. The work of addition can often be shortened by combining such of the quantities as can be readily united, and then adding, in a similar way, these results and the remaining quantities.

4. In the addition of mixed quantities, the integral and fractional parts can be added separately; or the mixed quantities can be first reduced to the fractional form and then added.

5. The final result should be reduced to its simplest form.

Illustrative Example.

Simplify $\frac{1}{4-x}+\frac{1}{7-x}+\frac{5x-29}{x^2-11x+28}$

$$+\frac{2(x+12)}{x^2-16}-\frac{18-x}{x^2-3x-28.}$$

Adding the first and second terms we have:

$$\frac{1}{4-x}+\frac{1}{7-x}=\frac{7-x}{(4-x)\,(7-x)}$$

$$+\frac{4-x}{(4-x)\,(7-x)}=\frac{11-2x}{(4-x)\,(7-x)}$$

Adding this result and the third term we have:

$$\frac{11-2x}{(4-x)\,(7-x)}=\frac{11-2x}{(x-4)\,(x-7)}$$

[illegible]

$$\frac{5x-29}{x^2-11x+28}=\frac{5x-29}{(x-4)\,(x-7)}$$

Adding $$=\frac{3x-18}{(x-4)\,(x-7)}$$

Combining this result with the remaining two terms we have:

$$\frac{3x-18}{(x-4)\,(x-7)}=\frac{3x^2-6x-72}{(x-4)\,(x-7)\,(x+4)},$$

$$\frac{2(x+12)}{x^2-16}=$$

$$\frac{2(x+12)}{(x-4)\,(x+4)}=\frac{2x^2+10x-168}{(x-4)\,(x-7)\,(x+4)},$$

$$-\frac{18-x}{x^2-3x-28}=$$

$$\frac{x-18}{(x-7)(x+4)}=\frac{x^2-22x+72}{(x-4)(x-7)(x+4)},$$

Adding $$=\frac{6x^2-18x-168}{(x-4)(x-7)(x+4)}$$

$$=\frac{6(x^2-3x-28)}{(x-4)(x-7)(x+4)}$$

$$=\frac{6(x-7)(x+4)}{(x-4)(x-7)(x+4)}=\frac{6}{x-4}\text{ Ans.}$$

EXAMPLES.

Simplify the following:

1. $\frac{ac}{b}+\frac{ab}{c}-\frac{bc}{a}$.

2. $\frac{a}{x}-\frac{x+a}{y}$.

3. $\frac{a}{a+3}+\frac{3}{a-3}$.

4. $\frac{5a}{a^2-25}-\frac{a}{a-5}$.

5. $\frac{1}{x-2}-\frac{1}{x-5}$.

6. $\frac{a+1}{1+a+a^2}+\frac{a-1}{1-a+a^2}$.

7. $\frac{3-2b}{5}+\frac{5-7b}{10}+\frac{8-2b}{25}$.

8. $\frac{2x-3y}{2(x-y)}+\frac{3x-2y}{3(x-y)}+\frac{6x-5y}{6(x-y)}$.

9. $$\frac{1}{1-a}+\frac{1}{1+a}-\frac{1}{1+a^2}$$

10. $$\frac{a-5}{a-6}+\frac{a-6}{a-5}-\frac{1}{(a-6)(a-5)}.$$

11. $$\frac{2}{x^2-4}+\frac{1}{x^2-6x+8}-\frac{4}{x^2-2x-8}.$$

12. $$\frac{a+3}{(a+1)(a+2)}-\frac{a+1}{(a+2)(a+3)}.$$

13. $$\frac{2}{(a+3)^2-4}+\frac{2}{(a+4)^2-9}.$$

14. $$\left(2x-y+\frac{x}{x-y}\right)-\left(x+y+\frac{y}{x+y}\right).$$

15. $$\frac{1-x}{1+x}-\frac{1+x}{1-x}+\frac{5x}{1-x^2}-\frac{x}{1+x^2}.$$

16. $$\frac{x-2y}{x^2+3xy+2y^2}+\frac{y}{x^2+4xy+3y^2}.$$

17. $$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$$

18. $$\frac{a^2-b}{(a-b)(a-1)}+\frac{b^2+a}{(b+1)(b-a)}+\frac{1+ab}{(1-a)(1+b)}.$$

19. $$\frac{2(1-3x)}{(1+x)(1+9x)}-\frac{1-2x}{(1+x)(1+4x)}+\frac{2}{1+4x}.$$

20. $\dfrac{a}{x^2-a^2}-\dfrac{1}{2}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right).$

21. $\dfrac{1}{x-1}-\left(\dfrac{1}{2(x+1)}+\dfrac{x+3}{2(x^2+1)}\right).$

22. $\dfrac{1}{x^3}+\dfrac{1}{x^2}+\dfrac{x-1}{x^2+1}-\left(\dfrac{1}{x}+\dfrac{1}{(x^2+1)^2}\right).$

23. $\dfrac{x}{x+2}-\dfrac{x^2}{x^2-x-6}+\dfrac{2x^2+8x-10}{x^3+4x^2-11x-30}.$

24. $\dfrac{b^2+ax}{x^2-b^2}-\left(\dfrac{b-a}{x-b}-\dfrac{a-2b}{b+x}-\dfrac{3x(a-b)}{b^2-x^2}\right).$

25. $3ax-4b^2y-\dfrac{ab}{a^2-b^2}-$
$\left(2ax+3by^2+\dfrac{ab}{(a-b)^2}\right).$

26. From $\dfrac{2}{x^2+14x+48}$ subtract $\dfrac{1}{x^2+13x+42}.$

27. From $\dfrac{1}{1-3a}$ subtract $\dfrac{2}{1+3a}+\dfrac{1-9a}{1-9a^2}.$

28. From $\dfrac{1}{x^2-y^2}$ subtract $\dfrac{1}{(x+y)^2}-\dfrac{1}{(x-y)^2}.$

29. From $x+y-\dfrac{3x}{x-y}$ subtract $3x-2y$
$-\dfrac{2y}{x+y}.$

30. From $\dfrac{(m+n)^2}{4mn}$ subtract 1.

MULTIPLICATION OF FRACTIONS.

125. To multiply a quantity by 6 and divide the result by 2, is the same as to multiply the quantity by $\frac{6}{2}$ or 3; or, using letters, to multiply any quantity, as the fraction $\frac{a}{b}$, by m and then to divide the result by n, is equivalent to multiplying the fraction by the quotient $\frac{m}{n}$. But we can multiply a fraction by multiplying its numerator (Prin. I.) and divide a fraction by multiplying its denominator (Prin. II.).

Hence $\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}$.

The same result is neatly obtained thus,

$$\frac{a}{b} \times \frac{m}{n} = ab^{-1} \times mn^{-1} = amb^{-1}n^{-1} = \frac{am}{bn}.$$

Factors common to the numerator and denominator can evidently be cancelled, without affecting the value of the product.

Hence the

RULE.

Indicate the multiplication and cancel all factors common to the numerator and denominator of the indicated product; multiply together the remaining factors of the numerator for a new numerator and the remaining factors of the denominator for a new denominator.

Remarks.

1. In the use of cancellation, the student should always have in mind the principle upon which it is based. He will thus avoid errors and save himself much useless labor.

2. It is of the utmost importance that all common factors of the numerator and denominator be cancelled before performing the multiplication. To insure this, it is better, in difficult examples, to write the terms of the given fractions in their factored form.

3. Polynomials containing fractional terms are sometimes, without any change of form, multiplied together under the general rule for the multiplication of polynomials.

4. Unless treated as stated in (3), entire and mixed quantities must be reduced to the fractional form before multiplication.

Illustrative Examples.

Multiply together $\frac{x^2+x+1}{x^6-1}$, $\frac{(x^2+2)^2-9x^2}{2x+5}$, $\frac{x^2-x+1}{x^2-4}$ and $2-\frac{3}{x+4}$.

$$\frac{x^2+x+1}{x^6-1}\times\frac{(x^2+2)^2-9x^2}{2x+5}\times\frac{x^2-x+1}{x^2-4}\times 2-\frac{3}{x+4}$$

$$=\frac{\cancel{x^2+x+1}}{\cancel{(x+1)}\cancel{(x^2-x+1)}\cancel{(x-1)}\cancel{(x^2+x+1)}}$$

$$\times\frac{\cancel{(x-2)}\cancel{(x-1)}\cancel{(x+2)}\cancel{(x+1)}}{\cancel{2x+5}}$$

$$\times\frac{\cancel{x^2-x+1}}{\cancel{(x-2)}\cancel{(x+2)}}\times\frac{\cancel{2x+5}}{x+4}=\frac{1}{x+4}\text{ Ans.}$$

Multiply together $\frac{1}{x^2}+\frac{1}{x}+1$ and $\frac{1}{x^2}-\frac{1}{x}+1$.

$$\begin{array}{l}
\frac{1}{x^2}+\frac{1}{x}+1 \\
\frac{1}{x^2}-\frac{1}{x}+1 \\
\hline
\frac{1}{x^4}+\frac{1}{x^3}+\frac{1}{x^2} \\
\qquad -\frac{1}{x^3}-\frac{1}{x^2}-\frac{1}{x} \\
\qquad\qquad +\frac{1}{x^2}+\frac{1}{x}+1 \\
\hline
\frac{1}{x^4}\qquad +\frac{1}{x^2}\qquad +1 \text{ Ans.}
\end{array}$$

EXAMPLES.

Multiply together the following:

1. $\frac{5ax^2}{6ay^3}$ and $\frac{4a^2y^5}{10x^3z}$.

2. $\frac{3mn^3}{2pq^2}$, $\frac{8px^3}{9my^2}$ and $\frac{15q^2y}{4n^2p^3}$.

3. $\frac{2x^2-x}{6}$ and $\frac{3}{2xy-y}$.

4. $\frac{a^2-2ab+b^2}{a+b}$ and $\frac{3}{a^2-b^2}$.

5. $\frac{x-y}{x^2+xy}$ and $\frac{x^2-y^2}{x^2-xy}$.

6. $\frac{9-z^2}{3+x}$, $\frac{9-x^2}{3z+z^2}$ and $1 + \frac{z}{3-z}$.

7. $\frac{x^2+3x}{x^2-3x-4}$, $\frac{x^2-16}{x^2-x}$ and $\frac{x^2+x}{x^2+7x+12}$.

8. $\frac{16}{y^2}-1$ and $\frac{y+2}{y^2-2y-8}$.

9. $\frac{a^2-a-2}{a^2-a-12}$, $\frac{a^2-4a-21}{a^2-7a-8}$ and $\frac{a^2-4a}{a^2-2a}$.

10. $\frac{a^2-c^2}{a^2-3ac+2c^2}$, $\frac{ac-2c^2}{a^2+ac}$ and $c-a-\frac{ac-c^2}{a}$.

11. $\frac{m^2-2mn+n^2-p^2}{m^2+2mn+n^2-p^2}$, $\frac{m^2-2pm+p^2-n^2}{m^2+2pm+p^2-n^2}$ and $\left(\frac{m+n+p}{m-n-p}\right)^2$.

12. a^2+a+1 and $\frac{1}{a^2}-\frac{1}{a}+1$.

13. $\frac{a}{b}-\frac{b}{a}+\frac{c}{d}-\frac{d}{c}$ and $\frac{a}{b}-\frac{b}{a}-\frac{c}{d}+\frac{d}{c}$.

14. $\frac{a^2+ab}{a^2+b^2}$ and $\frac{a}{a-b}-\frac{b}{a+b}$.

15. $\frac{9-a^2-b^2+2ab}{9+b^2-a^2+6b}$ and $\frac{9-a^2-b^2-2ab}{9+b^2-a^2-6b}$.

16. $\frac{5+a}{5-a}-\frac{5-a}{5+a}-\frac{4a^2}{25-a^2}$ and $\frac{5+a}{2a}$.

17. $\frac{3a^2-3x^2+a-x}{2}$ and $1+\frac{x}{a-x}$.

18. $\dfrac{2x+y}{x+y} - 1 - \dfrac{y}{y-x} - \dfrac{x^2}{x^2-y^2}$ and $\dfrac{x^2-y^2}{x^2+y^2}$.

19. $\dfrac{x}{yz} - \dfrac{y}{xz} - \dfrac{z}{xy} - \dfrac{2}{x}$ and $1 - \dfrac{x+y}{x+y+z}$.

20. $\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{a}$ and $\dfrac{x}{y} - \dfrac{y}{z} + \dfrac{z}{a}$.

DIVISION OF FRACTIONS.

126. $$\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn},$$

consequently $$\frac{am}{bn} \div \frac{m}{n} = \frac{a}{b},$$

but $$\frac{am}{bn} \times \frac{n}{m} = \frac{a}{b},$$

or dividend × inverted divisor = quotient.

Hence the

RULE.

To divide one fraction by another, invert the divisor and proceed as in multiplication.

Remarks.

1. The divisor must be reduced to a single, simple fraction before inversion.

2. In general, the dividend is also thus reduced; if not, it is treated as a polynomial and each of its

terms multiplied by the fractional divisor, inverted.

3. In certain examples the rule for the division of one polynomial by another is employed; the rule for the division of fractions being used, in determining the successive terms of the quotient.

Illustrative Examples.

1. Divide $9+\frac{5y^2}{x^2-y^2}$ by $3+\frac{5y}{x-y}$.

$$9+\frac{5y^2}{x^2-y^2}\div 3+\frac{5y}{x-y}=\frac{9x^2-4y^2}{x^2-y^2}\div\frac{3x+2y}{x-y}=$$

$$\frac{9x^2-4y^2}{x^2-y^2}\times\frac{x-y}{3x+2y}=\frac{(3x-2y)(3x+2y)}{(x+y)(x-y)}$$

$$\times\frac{x-y}{3x+2y}=\frac{3x-2y}{x+y}.$$

2. Divide $x^2+\frac{1}{x^2}+2$ by $x+\frac{1}{x}$.

$$\begin{array}{r|l} x^2+2+\frac{1}{x^2} & x+\frac{1}{x} \\ \cline{2-2} \underline{x^2+1\phantom{+\frac{1}{x^2}}} & x+\frac{1}{x} \\ \cline{2-2} 1+\frac{1}{x^2} & \\ \underline{1+\frac{1}{x^2}} & \end{array}$$

EXAMPLES.

1. Divide $\frac{6a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^2y^5}$.

2. Divide $\frac{x^2 - 9}{15}$ by $\frac{x + 3}{5}$.

3. Divide $\frac{x^2 - y^2}{x^2 - 2xy + y^2}$ by $\frac{x^2 + xy}{x - y}$.

4. Divide $\frac{1}{a^2 + 2a - 15}$ by $\frac{1}{a^2 - 2a - 3}$.

5. Divide $\frac{x^3 - 25x}{x^2 + 8x + 15}$ by $\frac{x^2 - 7x + 10}{x^2 + 5x + 6}$.

6. Divide $4x^2 - \frac{3yz}{2a}$ by $x - \frac{y - z}{a}$.

7. Divide $a^2 + \frac{a^3}{x - y}$ by $\frac{xy}{x - y} - a$

8. Divide $\frac{a^2}{b^3} - \frac{1}{a}$ by $\frac{a}{b^2} + \frac{1}{b} + \frac{1}{a}$.

9. Divide $\frac{a}{b} - \frac{5a - 3b}{a + b}$ by $\frac{a}{b} - \frac{3a - b}{a + b}$.

10. Divide $x^3 + x^2 - 1 - \frac{1}{x}$ by $x - \frac{1}{x}$.

11. Divide $\frac{x^2 + (m + n)x + mn}{x^2 + (m + p)x + pm}$ by $\frac{x^2 - n^2}{x^2 - p^2}$.

12. Divide $x^3 + \frac{1}{x^3}$ by $x + \frac{1}{x}$.

13. Divide $\frac{x^2 - y^2 - z^2 - 2yz}{x^2 - y^2 - z^2 + 2yz}$ by $\frac{x + y + z}{x + y - z}$.

14. Divide $\frac{a^2 + b^2 - c^2 - d^2 + 2ab - 2cd}{a^2 + c^2 - b^2 - d^2 + 2ac - 2bd}$ by

$$\frac{a + b - c - d}{a - b + c - d}.$$

15. Divide $1 + \dfrac{4ab}{a^2 + b^2 - c^2 - 2ab}$ by

$$2 - \frac{a - 3b - 3c}{a - b - c}.$$

16. Divide $x^3 + \dfrac{x^2}{y} + \dfrac{x}{y^2} + \dfrac{1}{y^3}$ by $\dfrac{x^2}{y}$.

COMPLEX FRACTIONS.

127. A complex fraction may be regarded as the indicated division of one fraction by another. It can be simplified by performing the division.

Remarks.

1. The dividing line of a complex fraction must be indicated with great care. An error in the choice of this line will change the value of the expression entirely, since it will change both numerator and denominator of the complex fraction.

The expressions $\cfrac{a+b}{\cfrac{c+d}{m+n}}$ and $\cfrac{\cfrac{a+b}{c+d}}{m+n}$, for instance, are entirely different in value.

2. After reducing the numerator and denominator of a complex fraction, each, to a single, simple fraction, it is well to invert the denominator (divisor) and write the example as an example in multiplication. If either the numerator or denominator is an entire quantity, it should be written in the fractional form.

Thus, $\dfrac{a + \dfrac{b+c}{m}}{m+n} = \dfrac{\dfrac{am+b+c}{m}}{\dfrac{m+n}{1}} =$

$$\frac{am+b+c}{m} \times \frac{1}{m+n}.$$

3. Complex fractions can often be advantageously reduced to simple ones by multiplying both terms by the lowest common multiple of their denominators.

Thus, in the example given below, if we multiply both terms of the complex fraction by $(c+d)\,(c-d)$ we will have:

$\dfrac{1 + \dfrac{c-d}{c+d}}{1 + \dfrac{c+d}{c-d}} = \dfrac{c^2 - d^2 + (c-d)^2}{c^2 - d^2 + (c+d)^2}$, a simple fraction

which can then be reduced to its simplest form.

EXAMPLES.

Simplify the following:

1. $\dfrac{x + 6 + \dfrac{8}{x-3}}{x + 5 - \dfrac{9}{x-3}}$

2. $\dfrac{1 - \dfrac{x}{x-y}}{1 - \dfrac{x}{x+y}}$

3. $$\frac{\dfrac{a+x}{b+y}+\dfrac{a-x}{b-y}}{\dfrac{a+x}{b-y}+\dfrac{a-x}{b+y}}$$

4. $$\frac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}$$

5. $$\frac{\dfrac{a+3}{a-3}+\dfrac{a-3}{a+3}}{\dfrac{a+3}{a-3}-\dfrac{a-3}{a+3}}$$

6. $$1-\frac{x}{1-x+\dfrac{x^2}{1+x}}$$

7. $$\frac{a+\dfrac{1}{b}}{a+\dfrac{1}{b+\dfrac{1}{c}}}-\frac{1}{b\,(abc+a+c)}$$

8. $$\frac{\dfrac{2x}{3}-\dfrac{x-1}{2}}{\dfrac{5}{3}(x+1)-\dfrac{3x}{2}-1\dfrac{1}{6}}$$

9. $$\frac{3}{x+1}-\frac{2x-1}{x^2+\dfrac{x}{2}-\dfrac{1}{2}}$$

10. $$\frac{\left(\frac{x}{a}+\frac{a}{x}\right)\left(\frac{a}{x}-\frac{x}{a}\right)}{1-\frac{x-a}{x+a}}$$

11. $$\frac{\frac{x^2+y^2}{x^2-y^2}+\frac{2x}{x+y}\left(\frac{xy-x^2}{(x-y)^2}+\frac{x+y}{x-y}\right)}{x-y}$$

12. $$\frac{\frac{(x^2-y^2)\,(2x^2-2xy)}{4\,(x-y)^2}}{\frac{xy}{x+y}}$$

13. $$\frac{\frac{ab}{x^2+(a+b)\,x+ab}-\frac{ac}{x^2+(a+c)\,x+ac}}{\frac{b-c}{x^2+(b+c)\,x+bc}}$$

When any term of a fraction contains a negative exponent, the fraction may be regarded as complex.

EXAMPLES.

Free the following from negative exponents:

1. $\dfrac{x^2+y^{-3}}{mn^{-4}e^{-2}}$ Ans. $\dfrac{e^2n^4x^2y^3+e^2n^4}{my^3}$

2. $\dfrac{1-2xy\,(x+y)^{-2}}{1+2xy\,(x-y)^{-2}}$

3. $\dfrac{1}{1+(1+x^{-1})^{-1}}$

4. $$\frac{m^2n^{-3} + m^{-1}}{mn^{-2} - (m - n)\,m^{-1}n^{-1}}$$

5. $$\frac{(a + b)\,(a - b)^{-1} + (a^2 + b^2)\,(a^2 - b^2)^{-1}}{(a - b)\,(a + b)^{-1} - (a^3 + b^3)\,(a^3 - b^3)^{-1}}$$

CHAPTER IX.

SIMPLE EQUATIONS.

GENERAL DEFINITIONS.

128. The symbols in an algebraic expression may be regarded as having any value. If the symbols in two expressions be given such values as to make the value of the expressions themselves the same, the two expressions may be placed equal to each other and an *equation* be thus formed.

An **equation**, then, is the symbolic statement that two algebraic expressions are equal.

Thus, $x + 3a = 5 + b$ is an equation.

129. The equal expressions are called the **members** of the equation. The expression to the left of the sign of equality is the **first member.** The expression to the right of the sign of equality is the **second member.**

In the equation given above, $x + 3a$ is the first member and $5 + b$ is the second member.

130. In certain equations values, entirely independent of one another, can be assigned to the several symbols without destroying the equality.

Thus, in the equation $(a + b)(a - b) = a^2 - b^2$, any values may be given a and b and the equation remain true. The members of such equations can by reduction, be made identical. The equation is therefore called an *identical equation* or an *identity*.

An identity is then an equation which is *necessarily true*, that is, true for *all* values of the quantities involved.

In an identity the sign $\equiv$, read identically equal to, is often employed in the place of the usual sign of equality.

131. The symbols in most equations, however, are not independent of one another. A relation exists between them which is expressed by the form of the equation.

Thus, $a + b = c$ expresses algebraically the fact that the quantities a, b and c are so related that the sum of the first two equals the third, the value of one being thus determined by its relation to the other two.

In general, all the symbols except one in a single equation may have independent values assigned them. The value of this one must obviously be such that when substituted with the other values in the equation the equality will be preserved. This quantity is therefore dependent upon the others; its value is *conditional* upon theirs and the equation is called an *equation of condition*. Hence,

An equation of condition is an equation in which the value of one of the quantities involved is conditioned upon the values of the others; it is then only *conditionally true*.

Hereafter *equation* will be taken to mean *an equation of condition*.

132. Both **known** and **unknown quantities** appear in an equation.

Known quantities are represented by figures, or letters whose values are assumed as given.

133. **A numerical equation** is one in which the known quantities are represented solely by figures.

134. **A literal equation** is one in which the known quantities are represented by letters or by letters and figures.

135. The **unknown quantities** of an equation are the quantities whose values are to be determined. They are generally represented by the last letters of the alphabet. This, however, is not at all necessary and the student should accustom himself to the idea that an unknown quantity may be represented by any letter.

136. **The degree of an equation** is determined by the greatest number of unknown prime factors occurring in any one term after the equation has been reduced to such form that no unknown quantity appears in the denominator of any term and the exponents of all unknown quantities are positive integers.

Thus, $\frac{ax^2y}{2b} + \frac{3bx}{ac} + 7a = 10xy$ is an equation of the third degree since the greatest number of unknown prime factors in any one term is three.

137. **A simple equation**, is an equation of the *first degree;* an equation of the *second degree* is called a

quadratic equation; an equation of the *third degree* is called a **cubic equation.**

138. Since it is obvious from the nature of an equation that independent values may be assigned to all its symbols except one, and that this one must have a particular value, it follows:

1. That every equation must contain at least one unknown quantity.

2. That one equation is sufficient to determine the value of one unknown quantity only.

3. That if an equation contains more than one unknown quantity, other conditions than those expressed by the equation must be given before the unknown quantities can be determined.

It will be found hereafter that these other conditions are expressed by equations, there being in each case as many equations as there are unknown quantities.

SOLUTION OF SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

139. To solve an equation which involves one unknown quantity is to find a value of the unknown quantity that will satisfy it.

A value of the unknown quantity is said to *satisfy* an equation when the members of the equation, after the value of the unknown quantity has been substituted, can be reduced to identical expressions.

To verify or test a value of the unknown quantity is to find whether the value satisfies the equation.

The *value* of the unknown quantity must be expressed in terms of the known quantities, that is, it must be an expression containing known quantities only.

The value of the unknown quantity is called the **root** of the equation.

140. Axioms. The following axioms are used in the solution of equations:

1. If the same or equals be added to equals the sums will be equal.

2. If the same or equals be subtracted from equals the remainders will be equal.

3. If equals be multiplied by the same or equals the products will be equal.

4. If equals be divided by the same or equals (when the divisors are not zero) the quotients will be equal.

5. Like powers or like roots of equal quantities are equal.

The application of these axioms to the equal members of an equation is obvious. There is danger, however, that their employment in the solution of equations will come to be regarded as mere mechanical devices. This should be avoided. Through the mechanism, the axioms should always be clearly seen. They will thus serve as guides pointing the way to correct results.

141. In solving a simple equation the aim is to reduce the equation to the form

$$x = A$$

that is, to an equation whose first member is the unknown quantity and whose second member is composed of known symbols.

To effect this reduction, it is necessary, in general, to perform the following operations:

1. To transpose terms.

2. To clear of fractions.

3. To make the coefficient of the unknown quantity unity.

142. These operations are based upon the following principles which in turn rest upon the axioms given:

Principle I. *To transpose a term from one member of an equation to the other, add the term with a contrary sign to both members; or, what is equivalent, transfer the term and change its sign.*

Thus, transpose a and $-b$ in the equation $a+x-b=c$. Adding $-a$ and $+b$ to each member which is the same as to subtract a and add b, (axioms 1 and 2) we have $x=c-a+b$.

The same result is obtained by transferring a and $-b$ from the first to the second member and changing the sign of each of these terms. Hence the principle.

Since all the terms of an equation can be transposed, it follows as a corollary of this principle, that the signs of all the terms of an equation can be changed at any time.

The same result is reached by multiplying both members of the equation by -1.

Principle II. *To clear an equation of fractions multiply each member by the L. C. M. of the denominators of the fractional terms.*

Thus, clear of fractions $\frac{3x}{4}+\frac{a}{2}+\frac{c}{6}=\frac{d}{12}$

Multiplying each member by 12, the L. C. M. of the denominators, we have (axiom 3) $9x+6a+2c=d$

Principle III. *To make the coefficient of the unknown quantity unity, divide both members of the equation by the coefficient of the unknown quantity.*

Thus, make the coefficient of x unity in the equation $(a+b)\,x=c$.

Dividing both members by $a+b$, we have (axiom 4)

$$x=\frac{c}{a+b}$$

By the aid of the foregoing principles, and the performance of indicated operations whenever convenient, all simple equations containing one unknown quantity may be reduced to the desired form:

$$x=A$$

in which the value of the unknown quantity is made manifest.

We may take then the following as the general rule for solving any simple equation with one unknown quantity.

RULE.

1. *Reduce each term to its simplest form; cancel factors common to the two members, when readily discernible and composed of known quantities only; collect terms whenever convenient.*

2. *Clear the equation of fractions.*

3. *Transpose the unknown terms to the first member of the equation, and the known terms to the second member.*

4. *After reducing each member to its simplest form, resolve the first member, or conceive it to be resolved, into two factors, one of which shall be the unknown quantity.*

5. *Divide both members of the equation by the coefficient of the unknown quantity, and reduce the result to its simplest form.*

It will be found convenient in many examples to vary the order given in the general rule. Other steps than those mentioned can sometimes be taken to advantage. The student must always bear in mind the ultimate purpose of his work and direct all his thought to such application of algebraic principles as will secure the desired result by the shortest and most elegant method.

The illustrative examples and remarks on particular processes should be studied with care before the general examples are taken up. Both should be carefully reviewed after some experience has been gained in the solution of examples.

143. Illustrative Examples.

1. Solve the equation

$$\frac{\frac{2}{7}x - 5}{3} + \frac{2x - 21}{7} - \frac{2x - 5}{3x - 4} = \frac{8x}{21} - 3.$$

Placing the denominator 7 under each term of the second numerator, we have,

$$(2) \qquad \frac{\frac{2}{7}x - 5}{3} + \frac{2x}{7} - 3 - \frac{2x - 5}{3x - 4} = \frac{8x}{21} - 3$$

Collecting terms and partially clearing of fractions by multiplying by 21

(3) $$2x - 35 + 6x - \frac{21\,(2x - 5)}{3x - 4} = 8x$$

Cancelling terms and dividing by -7,

(4) $$5 + \frac{6x - 15}{3x - 4} = 0$$

Clearing of fractions,

(5) $$15x - 20 + 6x - 15 = 0$$

Transposing terms,

(6) $$21x = 35$$

(7) $$x = \tfrac{5}{3}$$

2. Solve the equation $$\frac{1}{2\,(x - a)} - \frac{1}{3\,(x + a)} - \frac{x + 5a}{6\,(x^2 + a^2)} = \frac{a^3}{x^4 - a^4}$$

(1) $$\frac{1}{2\,(x - a)} - \frac{1}{3\,(x + a)} - \frac{x + 5a}{6\,(x^2 + a^2)} = \frac{a^3}{x^4 - a^4}.$$

Adding the first two terms,

(2) $$\frac{x + 5a}{6\,(x^2 - a^2)} - \frac{x + 5a}{6\,(x^2 + a^2)} = \frac{a^3}{x^4 - a^4}.$$

Clearing of fractions,

(3) $$(x + 5a)(x^2 + a^2) - (x + 5a)\,(x^2 - a^2) = 6a^3.$$

Collecting the coefficients of $(x + 5a)$,

(4) $$2a^2\,(x + 5a) = 6a^3.$$

Dividing by $2a^2$,

(5) $$x + 5a = 3a.$$

(6) $$x = -2a.$$

3. Solve the equation $$\frac{3x - 8}{x - 1} - \frac{2x - 11}{x - 3} = \frac{5x - 22}{x - 4} - \frac{4x - 38}{x - 9}.$$

(1) $$\frac{3x-8}{x-1}-\frac{2x-11}{x-3}=\frac{5x-22}{x-4}-\frac{4x-38}{x-9}.$$

Dividing each numerator by its denominator,

(2) $$3-\frac{5}{x-1}-\left(2-\frac{5}{x-3}\right)=5-\frac{2}{x-4}-\left(4-\frac{2}{x-9}\right).$$

Removing parentheses and cancelling same quantities in both members,

(3) $$-\frac{5}{x-1}+\frac{5}{x-3}=-\frac{2}{x-4}+\frac{2}{x-9}.$$

Adding fractions in each member of the equation,

(4) $$\frac{10}{(x-1)(x-3)}=\frac{10}{(x-4)(x-9)}.$$

Dividing by 10 and clearing of fractions,

(5) $$x^2-13x+36=x^2-4x+3.$$

Transposing and collecting terms,

(6) $$-9x=-33.$$

(7) $$x=\tfrac{11}{3}.$$

Remarks on Examples.

1. Two methods of dealing with the negative apparent sign of a fraction have been already explained (¶ 122). These methods, together with the principles upon which they are based, should be carefully recalled to mind.

2. No neat artifice for shortening the solution of an equation should be overlooked. The following are

the principal devices employed, each of which is exemplified in the illustrative examples:

(a) Cancelling, during the progress of the work, factors common to the two members of the equation.

(b) Uniting terms before clearing of fractions, providing the terms can be readily united.

(c) Clearing the equation of fractions by successive steps, clearing it of the simpler denominators first, and collecting the integral terms when possible.

(d) Indicating the multiplication whenever there is a probability of this step leading to the discovery of a factor common to the two members of the equation.

(e) Writing the denominator of a fraction under each term of the numerator before collecting terms.

(f) Reducing the fractions to their lowest terms or to mixed quantities.

3. An equation which appears to be of a higher degree than the first, is sometimes reducible to the first degree through simplifying the terms of the equation or by the elimination of the terms of higher degree during the process of reduction.

4. The successful student must learn to distinguish known from unknown quantities, discern and apply principles and employ concise methods of operation under different and confusing conditions. It is in part for this reason that examples are introduced in which the unknown quantities are represented by other than the last letters of the alphabet. These examples also prepare the way for similar work to be hereafter required of the student.

5. In preparing work for explanation in the class room, the student should obey implicitly the following directions:

(a) Number each equation.

(b) Avoid the common error of connecting by the sign of equality expressions that are unequal.

(c) Arrange the equations neatly, in order, with the work sufficiently elaborated so that the steps taken can be readily understood.

(d) Consider carefully the statement to be made of the steps taken in passing from one equation to another, and determine from their importance the equations to be read in full, and those which need to be referred to by number only or perhaps omitted entirely in the explanation.

EXAMPLES.

1. $5 + \frac{x-3}{4} = \frac{9-x}{12} + \frac{35}{6}$; find x.

2. $b - \frac{c-x}{b} = c - \frac{b-x}{c}$; find x.

3. $1 - (x-3)^2 + 5(x+4) = 26 - x[5 - (2-x)]$; find x.

4. $x + 3\frac{5}{6} - \frac{2x - (4 - 7x)}{6} = \frac{x}{3} - x\left(\frac{2}{3} - \frac{4}{x}\right)$; find x.

5. $\left(\frac{c}{2} - 1\right)\left(x - \frac{d}{3}\right) - \left(\frac{c}{3} - 2\right)\left(x - \frac{d}{4}\right) + \left(\frac{c}{4} - 1\right)\left(x - \frac{d}{5}\right) = 0$; find x;

6. $\dfrac{1}{x-3} = \dfrac{4 + \dfrac{x+1}{3-x}}{5}$ find x.

7. $\dfrac{x^2 + ax + b}{x + a} - c = x - \dfrac{cx - bc + a}{x - b}$; find x.

8. $6\left(2a - \dfrac{x}{3}\right)\left(3a - \dfrac{x}{2}\right) = 4x\left(\dfrac{x}{2} + a\right) -$
$\dfrac{1}{2}\left(10a - 5x\right)\left(a - \dfrac{2x}{5}\right)$; find x.

9. $\left(\dfrac{x^2 + nx}{x^2 + nx + n^2}\right)\left(\dfrac{x^3 - n^3}{nx\,(x + n)}\right) = 1$; find x.

10. $4a - [3a - 4\,\}10 - 2a - (\overline{5 - 6a} - 15)\{\,]$
$= 11$; find a.

11. $\dfrac{2x + 5\frac{1}{3}}{5 - 6x} - 3x - \dfrac{1 - 15x}{6} = 2\frac{2}{3} - \dfrac{15 + 9x}{18}$;
find x.

12. $\dfrac{1}{2 - \dfrac{3}{4 + \dfrac{5}{6 + x}}} = \dfrac{3}{4}$; find x.

13. $\dfrac{2}{x^2 - x - 2} - \dfrac{1}{x^2 + x - 2} - \dfrac{1}{x^2 + 3x + 2} = 0$:
find x.

14. $\dfrac{x^2 + (m + n)\,x + mn}{x^2 + (m - p)\,x - mp} = 3$; find x.

15. $$\frac{x - \frac{3(x-5)}{4}}{8} - \frac{x-3}{4} = x + 1 - \frac{8x - \frac{2(x+16)}{17}}{6}; \text{ find } x.$$

16. $$\frac{1}{(x-a)(x-b)} + \frac{1}{(b-x)(x-c)} = \frac{1}{(c-x)(d-x)} + \frac{1}{(a-x)(x-d)}; \text{ find } x.$$

17. $[1-(1+x^{-1})^{-1}]^{-1} = 1$; find x.

18. $$\frac{b-m}{b-m-1} - \frac{b-m-1}{b-m-2} = \frac{b-m-2}{b-m-3} - \frac{b-m-3}{b-m-4}; \text{ find } b.$$

19. $$\frac{1}{5ax+5ac} - \frac{1}{4bx-4bc} = \frac{1}{x^2-c^2}; \text{ find } x.$$

20. Solve 19 for a, when b, c, and x are known.

21. Solve 19 for b, when a, c, and x are known.

22. Solve 19 for c, when a, b, and x are known.

23. $$\frac{5}{\frac{m(m+x)^2}{m^3-mx^2} - 3} = \frac{2m}{x} - 1\frac{1}{4}; \text{ find } x.$$

24. $$ax^{-1} + (ax^{-1} - bx^{-1})(b-a)^{-1} = bx^{-1} - a^{-1}(a-b) - x^{-1}; \text{ find } x.$$

25. $$\frac{m}{5} - \frac{\frac{1}{3}(2m-3) - \frac{1}{8}(4m-2)}{\frac{1}{6}(m-4)} = \frac{2}{5}\left(\frac{m^2}{2m+10}\right); \text{ find } m.$$

26. $\dfrac{x - \frac{1}{6}}{x - \frac{1}{3}} - \dfrac{x - \frac{1}{2}}{x - \frac{2}{3}} = \dfrac{x - \frac{5}{6}}{x - 1} - \dfrac{x - \frac{7}{6}}{x - \frac{4}{3}}$; find x.

27. $\dfrac{6x - 11a}{x - 2a} - \dfrac{2x - a}{x - a} = \dfrac{7x - 29a}{x - 4a} - \dfrac{3x - 16a}{x - 5a}$; find x.

28. $\dfrac{1}{3 - x} + \dfrac{1}{4 - x} + \dfrac{3x - 11}{x^2 - 7x + 12} - \dfrac{2x}{x^2 - 9} = \dfrac{1}{2}$; find x.

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CHAPTER X.

PROBLEMS LEADING TO SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

144. In a problem the relation of the unknown to the known quantities is expressed in ordinary language. The student is asked to express this relation in algebraic equations and by their solution determine the value of the unknown quantities.

145. The unknown quantities in a problem are the coefficients of the *given* units. Thus it is the *number* of books, the *number* of days, the *number* of dollars, the *number* of some given unit, that is required. The algebraic symbolism employed in the solution pertains to the abstract coefficients. The operations upon these symbols are operations upon abstract quantities. The result is an abstract number. It is, however, the coefficient of a given unit and must be joined to that unit in order to be rightly interpreted. The interpreted result is the ascertained coefficient times the given unit.

146. To solve a problem is to find the value of the unknown quantity involved. The solution naturally separates itself into three steps:

I. Symbolizing the quantities dealt with.

II. Expressing the given relations between the known and unknown quantities in terms of equations.

III. Solving the equations.

In the problems considered in this chapter only one of the quantities involved is unknown, or if there are more than one, they are so related as to permit of their being readily expressed in terms of some one unknown quantity.

147. I. Expressing the quantities by symbols. After a thorough study of the relation of the quantities to each other, the student should choose as the fundamental or independent unknown quantity such one as stands in the simplest relation to the others to be considered. It is sometimes advisable to choose as this quantity one not directly mentioned in the problem, but one which is so related to each of those to be treated as to permit their values to be readily expressed in terms of this basal quantity. This quantity is generally represented by one of the last letters of the alphabet, as x.

The following examples give the student practice in choosing this basal quantity, and in deriving from it expressions for the quantities whose relation to it are known.

EXAMPLES.

1. The wheel of a carriage has a circumference of 5 feet. How many revolutions will it make in going a certain number of rods?

Let $x =$ the number of rods passed over.

$16\frac{1}{2}x =$ the number of feet passed over.

$\frac{16\frac{1}{2}x}{5} = \frac{33x}{10} =$ the number of revolutions in going x rods.

2. A farmer paid \$300 for a certain number of cows; he sold them for \$5 apiece more than he gave; what was the selling price?

Let $x =$ the number of cows,

$\frac{300}{x} =$ number of dollars paid for one cow,

$\frac{300}{x} + 5 =$ number of dollars in the selling price of a cow.

3. A certain room is square; another is a feet longer and b feet narrower; what would express the difference in the areas of the two rooms?

Let $x =$ the number of feet in the side of the square room.

$x^2 =$ the number of square feet in the area of the square room.

$x + a =$ the number of feet in the length of second room.

$x - b =$ the number of feet in the width of the second room.

$x^2 + (a - b)x - ab =$ number of square feet in the area of second room.

$(a - b)x - ab =$ difference in area of the two rooms.

4. A man walks the distance between A and B at a fixed rate, in 5 hours. Find expressions for the distance and rate.

5. What must be added to any number to make p?

6. Find an expression for the rate at which a man will walk if he goes 20 miles in a certain number of hours.

7. I spent a number of dimes and 3 times as many quarters. How many cents have I left out of $4.

8. A boy spent m dollars more than half his money for a cap and then two-thirds of the remainder for a book. Find an expression for the amount of money he had left.

9. I loaned $75 at a certain rate per cent. What was the interest at the end of 4 years and 6 months?

10. I have n dollars which I divide among a certain number of people. If I divided m more dollars among a less people, what would each receive?

11. Find an expression for the area of a room which, if it were 2 feet shorter and 3 feet wider, would be square.

12. A crew rows at 3 times the rate of the stream. How long will it take to row 20 miles up stream?

13. Write an expression for a number whose digits increase by 1 from left to right.

14. Two men do a piece of work in c days. Find expressions for the part of the work done by each in 1 day.

15. If it takes m pieces of a certain kind of money to make a dollar, find an expression for the value of a fixed number of these pieces.

16. The minute hand and hour hand of a clock are together. Find expressions for the distances passed over by each hand at a certain time.

17. One man walks a certain distance in 12 minutes, another in 10 minutes. How many rods does one gain over the other in 1 hour?

18. Two gamblers A and B have equal amounts. A wins from B the $\frac{1}{c}$th part of his money and then B gains from A the $\frac{1}{c}$th part of what he then has. Find expressions for what each has in the end.

148. II. Formation of the equation. The conditions given in the problem will always enable the student to build up two algebraic expressions for the same or equal things, which expressions united by the sign of equality will give the equation desired. In many problems the student has a choice as to what particular thing or what particular pair of equal things, among those given, he will use in forming the equation. To decide upon the quantities available, make the choice mentioned, and deduce correct expressions for the members of the equation, often demands of the student a high degree of patience, courage, ingenuity, and good judgment. Success requires a thorough knowledge of the quantities to be handled and their relation to each other. The equation must be thought out, not built up mechanically.

The following problems will illustrate the necessity of choosing at the beginning of the solution, the particular thing or pair of equal things, which expressed symbolically shall constitute the members of the equation; they also show that different things or different pairs of equal things, may be chosen and thus the same problem lead to several equations, any one of which, when solved, will give the correct answer to the problem.

Illustrative problems.

1. A man traveling has 50 minutes to wait for a train. How many blocks may he walk at the rate of 30 blocks in an hour in order to return by a street car which goes 4 times as fast, allowing 10 minutes to wait for a car?

If the members of the equation represent the time spent in walking and riding the following will be the solution of the problem:

Let $x =$ the number of blocks he can walk.
$2 =$ the number of minutes required to walk 1 block.
$2x =$ the number of minutes required to walk x blocks.
$\frac{1}{2} =$ the number of minutes required to ride 1 block.
$\frac{x}{2} =$ the number of minutes required to ride x blocks.
$2x + \frac{x}{2} =$ the number of minutes spent in riding and walking.
$40 =$ the number of minutes spent in riding and walking.

(1) $2x + \frac{x}{2} = 40$
(2) $5x = 80$
(3) $x = 16$

If the distance he walked and the equal distance he rode are fixed upon as the members, the solution will be as follows:

Let $x =$ the number of hours spent walking.
$30x =$ the number of blocks walked.
$\frac{2}{3} - x =$ the number of hours spent riding.
$(\frac{2}{3} - x)$ $120 =$ the number of blocks he rides.
(1) $30x = (\frac{2}{3} - x)\ 120$
(2) $30x = 80 - 120x$
(3) $150x = 80$
(4) $x = \frac{8}{15}$
$30x = 16$, the number of blocks walked.

2. Divide 84 into two such parts that the greater divided by the less will give 2 as a quotient and 12 as a remainder.

To this problem there are three solutions according as the members of the different equations represent different quantities.

1. Each member represents the greater part divided by the less.

2. Each member represents the greater part.

3. Each member represents the entire number.

1. Let $x =$ the lesser part.
$84 - x =$ the greater part.
$\frac{84 - x}{x} =$ result of greater divided by the less.

$2 + \frac{12}{x}$ = result of the greater divided by the less.

$$\frac{84 - x}{x} = 2 + \frac{12}{x}$$

2. Let x = the lesser part.

$84 - x$ = the greater part.

Divisor times quotient plus remainder equals dividend, therefore

$2x + 12$ = the greater part.

$84 - x = 2x + 12.$

3. Let x = the lesser part.

Then as above $2x + 12$ = the greater part,

$3x + 12$ = the number,

84 = the number,

$3x + 12 = 84.$

149. III. Solving the equation. This is done in accordance with the rules already explained.

150. Remarks.

1. While the solution of a problem is divided into parts, the problem should at first be studied as a whole. A clear perception of the quantities to be used in the solution and their relation to one another is necessary to a wise choice of the basal unknown quantity, but the quantities to be thus used cannot be fully decided upon until the question as to what the members of the equation shall represent has received consideration. Thus the student must have thought his way through to a mental mastery of the problem before taking up the first step in its written solution.

2. The student in his work should look forward to an equation which he can analyze. Simplicity in form should be sacrificed to simplicity in logical construction. The student should never be satisfied with an equation which he cannot explain.

3. Quantities of the same kind in an equation should be expressed in terms of the same unit. Distance for instance may be expressed in terms of feet, yards or miles, but only one unit should be used.

4. It is an excellent practice for the student to test the answer, obtained, by noting whether it will satisfy the conditions of the problem. The answer is, of course, simply the discovered value of the unknown quantity in the problem. If the same operations are performed upon the symbol representing this unknown quantity as are performed upon the answer in verifying it, the equation from which the answer is derived will be produced. This is in fact one of the desirable ways of forming such equation.

151. Additional illustrative examples.

1. A farmer sold a cow for a certain sum, a sheep for $\frac{1}{3}$ as much as the cow, and a horse for 4 times what he received for both the cow and the sheep. He received in all $160. For how much was each sold?

Let x = the number of dollars received for a cow,
then $\frac{1}{3}x$ = the number of dollars received for a sheep.
$\frac{4x}{3}$ = the number of dollars received for a cow and a sheep.
$\frac{16x}{3}$ = the number of dollars received for a horse.
$\frac{20x}{3}$ = the number of dollars received for all.
$160 = the number of dollars received for all.

first 5.

(1) $\frac{20x}{3} = 160.$
(2) $20x = 480.$
(3) $x = 24.$
$\frac{1}{3}x = 8.$
$\frac{16x}{3} = 128.$

2. Two men and a boy can do a piece of work in 5 days. One man does twice as much in a day as the boy and the other three times as much. In how many days can the boy alone do the work?

Let x = the number of days in which the boy alone can do the work.

then $\frac{1}{x}$ = part of the work which the boy does in one day.

$\frac{2}{x}$ = part of the work which one man does in one day.

$\frac{3}{x}$ = part of the work which the other man does in one day.

$\frac{6}{x}$ = part of the work which two men and the boy do in one day.

$\frac{1}{5}$ = part of the work which two men and the boy do in one day.

(1) $\frac{6}{x} = \frac{1}{5}.$
(2) $x = 30.$

PROBLEMS.

1. Divide 72 into two such parts that one-fourth of the first part may be equal to one-fifth of the second.

2. A boy had a certain sum of money. He spent one-tenth of it for marbles, one-fourth of it for a knife, and one-half of it for a book and found he had 45 cents remaining. How much had he?

3. A man employed three workmen. To the first he gave a dollars more than to the second; to the third b dollars less than to both the others. They all together received c dollars. How much was given to each?

4. A contractor paid his 5 workmen $2 per day and found he had $40 profit. If he had paid $3 a day he would have lost $30 on the transaction. How many days did it require to do the work?

5. A boy said to his sister, your age and mine together make 21 years; in 2 years you will be $\frac{2}{3}$ as old as I. How old was each?

6. A man has a note which was to run 16 years. $\frac{1}{3}$ of the time past, is 4 years less than $\frac{3}{5}$ of the time it still has to run. When will it be due?

7. A grocer mixed 3 kinds of tea. Of the first kind he had 4 pounds more than $\frac{1}{2}$ the whole; of the second, 1 pound less than $\frac{1}{3}$ the whole and of the third, 11 pounds less than $\frac{3}{4}$ the first kind. How many pounds were there of each?

8. A boy in playing with blocks finds that if he arranges them in a square, he has 4 left over, but that if he has 2 more in the length and one less in the width than he had before, the blocks will all be used. How many blocks had he?

9. A man bought one variety of apples at 6 for 5 cents and another variety at 5 for 7 cents. He sold them all at the rate of 3 for 4 cents and made 20 cents by the transaction. How many were there of

each kind, if there were 12 more of the second than of the first kind?

10. Having a certain sum of money, a man invested it as follows: $\frac{1}{4}$ of it and \$50 he invested in a city lot, with $\frac{1}{2}$ of the remainder and \$500 he built a house, and put $\frac{2}{3}$ of the remainder and \$200 in the bank which exhausted his money. How much had he at first?

11. A man has \$8.75. He has twice as many half dollars as dollars, three times as many quarters as halves, four times as many dimes as halves and quarters together, and as many nickels as all other pieces. How many has he of each?

12. A tank is being filled by three pipes. The first alone fills it in 2 hours, the second in 3 hours and the third in 3 hours and 36 minutes. How long would it require all to fill it?

13. A man has a house, a lot and \$300 in the bank. The house and the money in the bank are equal to $\frac{1}{3}$ the value of the lot; and the lot together with the money in the bank, equals $1\frac{1}{17}$ the value of the house. What is the value of the house and of the lot?

14. A man in going from his house to his place of business, 19 blocks away, stops first at a grocery and then at the post-office. The distance from his home to the grocery is to the distance from the grocery to the post-office in the ratio of 3 to 4; and $\frac{1}{6}$ of the distance from his home to the grocery is equal to $\frac{1}{5}$ of the distance from the post-office to his place of business. How far apart are the places?

15. My age now has the same ratio to my age a years ago that my age b years hence will have to my age b years ago. What is my age?

16. Find four numbers each larger by two than the next lower, such that the amount that the least is below 12, divided by the amount that the greatest is above 5, gives 1 as a quotient and 3 as a remainder.

17. The hour hand and minute hand of a clock are together at 12. When were they last together?

18. A man loaned a certain sum of money from which he gained 5%. He invested $\frac{2}{3}$ as much in an enterprise from which he lost 10%, and $1200 in government bonds at 4%. He found at the end of a year that his losses just equaled his gains. What sum had he loaned, and what sum invested in the losing enterprise?

19. A number is composed of three digits, of which the one in units place is twice the one in hundreds place, and the one in tens place is half the sum of the other two. If 178 is added to the number, a number will be formed which will have the tens digit of the original number in both units and hundreds place, and for a tens digit the hundreds digit of the original number. What is the number?

20. A man wishes to go by rail from one city to another, and return by a road which is one and one-half times the distance by rail. If he drives 8 miles an hour and the train goes 24 miles an hour, what is the distance by rail from one city to the other, if the round trip requires 1 hour and 50 minutes?

21. Divide b into 4 such parts that if the first is increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, all will be equal.

22. A man engaged to work a days at m dollars per day and his board which was reckoned at $1 per day. At the end of the a days he received n dollars having been obliged to pay his board the days he was idle. How many days was he idle?

23. A man has 4 wine casks, one full and three empty. He pours into the first empty cask, half the wine and half a gallon over; then into the second, one-third the remainder and one-third of a gallon over; and lastly into the third, one-fourth the remainder and one-fourth of a gallon over, when he has 11 gallons left in the original cask of wine. How much was in this cask at first?

24. A man has a certain amount of money invested. His expenses are a dollars per year. At the end of each year he adds to his stock $\frac{1}{3}$ of the amount of his money not thus expended, and at the end of 3 years finds his property is increased by one-third the original amount. How much had he at first?

25. A man invested an $\frac{a}{c}$th part of his money at $d\%$ and the remainder at $e\%$. His income is m dollars. How much money had he?

26. The hour hand, minute hand and second hand of a clock turn on the same center. The second hand and minute hand are together at 12 when the hour

hand points to 6. When will the second hand be half way between the other two?

27. A park is surrounded by a circular walk one and one-half miles in length. B's step is $\frac{2}{3}$ as long as A's, but A takes 4 steps while B takes 5. Starting together and walking in opposite directions, how many steps will each have taken when they meet, if A's step is two and one-half feet?

28. A, B, and C engage to do a piece of work. A alone can do it in m hours. A and B together in n hours. C does as much in a day as both A and B. A begins work at 8 in the morning and works until 12. B commences at 1 o'clock and works until relieved by C who finishes the work at 5 o'clock. When did C begin to work?

29. A dog is m rods behind a rabbit which is run- c miles an hour. At what rate per hour is the dog running if it catches the rabbit in 5 minutes?

30. A train leaving Chicago at 10 p. m. reaches Omaha at 2 p. m. the following day, while a train leaving Omaha at 5 p. m. reaches Chicago next morning at 8. How far from Chicago will the trains meet which leave Chicago and Omaha on the same evening, if the distance between Chicago and Omaha is 496 miles?

31. Two towns A and B are 16 miles apart. A man makes the round trip from A to B and return on a bicycle at the rate of 10 miles an hour, while another starts at the same time to drive from B to A and return at the rate of 6 miles an hour.

(a) Find the distance from A of the first point of meeting.

(b) Find the distance from A of the second point of meeting.

32. Two boats on a river are going in opposite directions. The one going down stream is 54 feet long and simply floats with the current, the other 78 feet long passes it in 18 seconds. What is the rate of the current and at what rate is the boat moving up stream, if the boat up stream is moving at one and one-half times the rate of the one down stream?

CHAPTER XI.

SIMULTANEOUS SIMPLE EQUATIONS.

152. We have found that any simple equation containing one unknown quantity can be solved and the value of the unknown quantity ascertained. Only one value can be found which will satisfy the equation; the equation can be said to determine this value.

153. An indeterminate equation is one in which the unknown quantities involved may have an indefinite number of values.

Take for example $3y - 4x = 20$.

If $x = 1$ $\quad y = 8$.

If $x = 4$ $\quad y = 12$.

If $x = 8$ $\quad y = 14$.

If $x = n$ $\quad y = \dfrac{20 + 4n}{3}$, in which any value may be assigned to n.

It is thus seen that the unknown quantities may have an infinite number of *sets of values* which will satisfy the equation. The same is true of any equation containing more than one unknown quantity. Such equations are therefore appropriately called *indeterminate*.

154. Let us consider the equations:

(1) $3x + 2y = 8$.

(2) $5x + 7y = 17$.

These equations are independent of each other in the sense that one cannot be derived from the other.

They can be put in the form,

$$(3) \quad y = \frac{8 - 3x}{2} \text{ and}$$

$$(4) \quad y = \frac{17 - 5x}{7}.$$

Assigning values to x the corresponding values of y in each equation can be obtained. If x be given the value 2, y will in each case equal 1. This is the only pair of values which will satisfy both equations; for it is evident that any value of x which will make y in the two equations equal, *must* make the second members of (3) and (4) equal or $\frac{8 - 3x}{2} = \frac{17 - 5x}{7}$. The only value of x which will satisfy this equation will be found upon solution to be 2, but $x = 2$ cannot give, in either of the original equations, any other value than $y = 1$. This pair ($x = 2, y = 1$) is therefore the only pair of values common to the two equations. It follows then that if the condition be imposed upon these two equations, that the values of the unknown quantities in the one shall be the same as in the other, these values are fixed and determinate. These equations are *independent simultaneaus equations.*

155. An independent equation is one which can not be derived from others in a given group by reduction or combination.

156. Simultaneous equations are those in which the values of the unknown quantities are limited to the set or sets which will satisfy all the equations.

157. The problem here considered is to find the value of the unknown quantities in a given group of independent simultaneous equations. It is evident that if from the equations given, one simple equation containing one unknown quantity can be deduced, the value of this unknown quantity can be found. It will be shown, directly, that by proper substitutions, the values of the other unknown quantities can then be readily discovered. It is clear that to reach the single equation desired, all the unknown quantities except one must be eliminated. In the illustrative example already considered, two equations were given containing two unknown quantities. The unknown quantity y was eliminated leaving a single equation containing x. One unknown quantity was made to disappear, and at the same time the number of equations to be dealt with was reduced by one.

Let us consider the case of three independent simultaneous equations involving three unknown quantities:

$$\begin{array}{ll} (1) & 2x + 2y - z = 17 \\ (2) & 3x - y - 3z = 8 \\ (3) & x - 3y + 5z = -2 \end{array}$$

$$\begin{array}{lll} \text{Multiplying (2) by 2,} & (4) & 6x - 2y - 6z = 16. \\ & (1) & \underline{2x + 2y - z = 17.} \\ \text{Adding (4) and (1),} & (5) & 8x - 7z = 33. \\ \text{Multiplying (2) by 3,} & (6) & 9x - 3y - 9z = 24. \\ & (3) & \underline{x - 3y + 5z = -2.} \\ \text{Subtracting (3) from (6),} & (7) & 8x - 14z = 26. \end{array}$$

Now considering (5) and (7), $\left\{\begin{array}{l} 8x - 7z = 33 \\ 8x - 14z = 26 \end{array}\right\}$

we have two equations containing two unknown quantities; that is, starting with three equations containing three unknown quantities, one unknown quantity (y) has been eliminated and the equations to be dealt with reduced in number from three to two. By treating these two remaining equations in the manner employed in the first illustrative example the unknown quantity x can be readily eliminated and the desired equation with its one unknown quantity (z) obtained.

As in a group of three, so in any group of independent simultaneous equations, combinations can be made which will eliminate one of the unknown quantities, and at the same time lessen by one the number of equations to be considered. If the number of unknown quantities involved is the same as the number of equations in the group, it follows that by the successive elimination of unknown quantities and the consequent reduction in the number of equations, there will always result a single equation containing one unknown quantity. It is evident that in such a group there will be but one set of values for the unknown quantities which will satisfy all the equations.

158. Elimination is the process of combining simultaneous equations so as to cause an unknown quantity to disappear.

There are three methods of elimination in common use:

I. By addition or subtraction.
II. By substitution.
III. By comparison.

SIMULTANEOUS EQUATIONS WITH TWO UNKNOWN QUANTITIES.

The methods of elimination will first be applied to simultaneous equations of two unknown quantities.

159. I. Elimination by addition or subtraction.

RULE.

1. *Multiply each equation by the least multiplier which will make the terms containing the unknown quantity to be eliminated, numerically identical.*

2. *Add or subtract the equations as necessary to cause these terms to disappear.*

Remarks on the solution of examples.

1. The value of the second unknown quantity can be found by substituting the value of the one first obtained, in any one of the equations employed in the solution. The simplest equation should be chosen.

2. It is often advisable to obtain the value of the second unknown quantity, not by substitution, but by eliminating the first unknown quantity from the original equations.

3. When the equations are fractional and the unknown quantity appears in the denominators, it is usually best to eliminate, before clearing of fractions.

4. Those terms should be eliminated which can be made numerically equal with the least difficulty. They are generally those containing the unknown quantity having the simpler coefficients.

5. The unknown quantity remaining, after the elimination, should appear in only one term of the resulting equation. Its coefficient, if a polynomial, should be inclosed in a parenthesis.

Illustrative Examples.

1. Solve the equations $\left\{\begin{array}{ll}(1) & 3x - 7y = 2. \\ (2) & 5x + y \ = 16.\end{array}\right.$

$$\begin{array}{lll}
\text{Multiplying (2) by 7,} & (3) & 35x + 7y = 112. \\
 & (1) & \underline{\ \ 3x - 7y = \ \ \ 2.} \\
\text{Adding (3) and (1),} & (4) & 38x = 114. \\
\text{Dividing by 38,} & (5) & x = 3. \\
\text{Substituting in (2),} & (6) & 15 + y = 16. \\
 & (7) & y = 1.
\end{array}$$

2. Solve the equations,

$$(1)\quad \frac{x}{m+n} + \frac{y}{m-n} = c,$$

$$(2)\quad \frac{x}{m-n} + \frac{y}{c} = m - n.$$

Clearing of fractions,

$$(3)\quad (m-n)\,x + (m+n)\,y = c\,(m+n)\,(m-n),$$

$$(4)\quad cx + (m-n)\,y = c\,(m-n)^2.$$

Multiplying (3) by c and (4) by $m-n$,

$$(5)\quad c(m-n)\,x + c\,(m+n)\,y = c^2(m+n)(m-n),$$

$$(6)\quad \underline{c(m-n)\,x + (m-n)^2\,y = c\,(m-n)^3}.$$

Subtracting,

$$(7)\quad [c(m+n) - (m-n)^2]\,y = c(m-n)\,[c(m+n) - (m-n)^2].$$

Dividing by the coefficient of y,

$$(8)\quad y = c\,(m-n).$$

Substituting this value of y in (3),

$$(9)\quad (m-n)\,x + c\,(m-n)\,(m+n) = c(m+n)\,(m-n).$$

$$(10)\quad (m-n)\,x = 0.$$

$$(11)\quad x = 0.$$

3. Solve the equations $\begin{cases} (1) & \dfrac{1}{x} - \dfrac{2}{y} = 2, \\ (2) & \dfrac{3}{x} + \dfrac{1}{y} = 20. \end{cases}$

Multiplying (1) by 3, (3) $\dfrac{3}{x} - \dfrac{6}{y} = 6.$

(2) $\dfrac{3}{x} + \dfrac{1}{y} = 20.$

Subtracting (2) from (3), (4) $-\dfrac{7}{y} = -14.$

(5) $-14y = -7.$

(6) $y = \frac{1}{2}.$

Eliminate y in the same manner:

(1) $\dfrac{1}{x} - \dfrac{2}{y} = 2.$

Multiplying (2) by 2, (7) $\dfrac{6}{x} + \dfrac{2}{y} = 40.$

Adding (1) and (7), (8) $\dfrac{7}{x} = 42.$

(9) $42x = 7.$

(10) $x = \frac{1}{6}.$

160. II. Elimination by substitution.

RULE.

Derive in either equation an expression for the value of the unknown quantity to be eliminated; substitute this expression for the unknown quantity in the other equation; the result will be a single equation containing only one unknown quantity.

Remarks on the solution of examples.

1. The student should carefully note that substitution in algebra is the replacing of one quantity by another *algebraically equal* to it; in other words, it is striking out the first quantity and putting in its place a second, of equal *algebraic* value. The student, overlooking the full significance of this operation, sometimes strikes out not only the first quantity but the sign preceding it, and thus it occurs that in completing his work he introduces a quantity numerically, but not algebraically, equal to what he has stricken out. Thus in the equations, $y = \frac{5 - x}{7}$ and $x = -2y$, the careless student will conceive $-x$ to be erased and $-2y$ put in its place, giving $y = \frac{5 - 2y}{7}$; when, to make the substitution correctly, he should erase only x, giving $y = \frac{5 - (-2y)}{7}$ or $y = \frac{5 + 2y}{7}$.

2. Remembering that the purpose of substitution in this work is to obtain a single equation containing only one unknown quantity, it will be found that this can sometimes be done to advantage by substituting the value of an entire expression rather than that of the unknown quantity itself. See second illustrative example.

Illustrative Examples.

1. Solve the equations $\begin{cases} (1) \; 3y - 4x = -10, \\ (2) \quad x + 5y = -9. \end{cases}$

Solving (2) for x, (3) $x = -9 - 5y$.

Substituting in (1), (4) $3y - 4(-9 - 5y) = -10$.

Removing parenthesis, (5) $3y + 36 + 20y = -10.$

(6) $23y = -46.$

(7) $y = -2.$

Substituting in (3) (8) $x = -9 + 10 = 1.$

2. Solve the equations $\begin{cases} (1) & \dfrac{a_1}{x - y + b} + \dfrac{c}{y} = c, \\ (2) & \dfrac{a}{y} - \dfrac{a_1}{x - y + b} = \dfrac{a}{b} - c + \dfrac{c}{b}. \end{cases}$

Solving (1) for $\dfrac{a_1}{x - y + b}.$

(3) $$\frac{a_1}{x - y + b} = c - \frac{c}{y}.$$

Substituting this value of $\dfrac{a_1}{x - y + b}$ in (2),

(4) $$\frac{a}{y} - \left(c - \frac{c}{y}\right) = \frac{a}{b} - c + \frac{c}{b}.$$

(5) $$\frac{a + c}{y} = \frac{a + c}{b}.$$

(6) $$y = b.$$

Substituting in (3), (7) $$\frac{a_1}{x} = c - \frac{c}{b}.$$

(8) $$x = \frac{a_1 b}{bc - c}.$$

161. III. Elimination by comparison.

RULE.

Derive from the two equations expressions for the value of one of the unknown quantities in terms of the other, and place the expressions thus derived equal to each other.

Note. It is sometimes best to solve the equations for identical expressions, as in the second illustrative example.

Illustrative Examples.

1. Solve the equations $\begin{cases} (1) & 3x-4y=3, \\ (2) & 5x+3y=34. \end{cases}$

Solving (1) for x, (3) $x=\dfrac{3+4y}{3}$.

Solving (2) for x, (4) $x=\dfrac{34-3y}{5}$.

Placing the values of x equal to each other, (5) $\dfrac{3+4y}{3}=\dfrac{34-3y}{5}$.

Clearing of fractions, (6) $15+20y=102-9y$.

Transposing and uniting, (7) $29y=87$.

(8) $y=3$.

Substituting y in (3), (9) $x=\dfrac{3+12}{3}=5$.

2. Solve the equations $\begin{cases} (1) & \dfrac{1}{x+1}-\dfrac{2}{x+y+1}=0, \\ (2) & \dfrac{3}{x+1}+\dfrac{2}{x+y+1}=2. \end{cases}$

Solving (1) for the expression $\dfrac{2}{x+y+1}$.

$$(3) \qquad \frac{2}{x+y+1}=\frac{1}{x+1}.$$

Solving (2) for the expression $\dfrac{2}{x+y+1}$,

$$(4) \qquad \frac{2}{x+y+1}=2-\frac{3}{x+1}.$$

Placing the two values of $\frac{2}{x+y+1}$ equal to each other,

$$(5)\qquad \frac{1}{x+1}=2-\frac{3}{x+1}.$$

Transposing and collecting,

$$(6)\qquad \frac{4}{x+1}=2.$$

Clearing of fractions,

$$(7)\qquad 4=2x+2.$$

$$(8)\qquad 2x=2.$$

$$(9)\qquad x=1.$$

Substituting in (3),

$$(10)\qquad \frac{2}{y+2}=\frac{1}{2}.$$

$$(11)\qquad 4=y+2.$$

$$(12)\qquad y=2.$$

162. The student should make himself familiar with the different methods of elimination by solving a number of examples under each method. In the majority of examples the method of addition and subtraction will be found to be most convenient.

163. Simultaneous equations with three or more unknown quantities.

It has been shown in article 157 that a number of equations containing the same number of unknown quantities can be so combined as to produce a second group of equations, one less in number and involving one less unknown quantity. This second group can be operated upon in a similar manner with a similar result, until by a series of such operations, one equation containing one unknown quantity can be derived. Hence the rule.

RULE.

1. *Decide upon the unknown quantity which can be most readily eliminated.*

2. *Decide upon the equation which can be most easily combined with the other equations.*

3. *Eliminate the unknown quantity between this and each of the other equations.*

4. *Deal in a similar manner with the new group of equations thus formed, and with the resulting groups in succession, until a single equation, involving one unknown quantity, is obtained. Solve this equation for the unknown quantity.*

5. *Substitute the value of this unknown quantity in the simplest of the equations containing two unknown quantities, and solve for the second unknown quantity. Substitute the values of the two quantities found in the simplest of the equations containing three unknown quantities, and solve for the third unknown quantity. Continue in a similar manner until the value of each of the unknown quantities is determined.*

164. Remarks on the solution of examples.

1. The student should notice that he can eliminate the unknown quantities in any order and use any equation as the eliminating equation.

2. The unknown quantity chosen for elimination should be, in general, the one which appears in the least number of equations, or has the simplest coefficients.

3. The equation chosen for combination with the others, should be the one through whose use the elimination can be effected by the simplest multiplications or the least number of operations. It sometimes oc-

curs that by a wise selection of such equations, two unknown quantities, can be eliminated by a single combination.

4. It is a helpful drill for the student to select the unknown quantity to be eliminated, decide upon the eliminating equation, note the combinations to be made, and make record of the unknown quantities contained in each of the resulting equations. The steps to be taken in the solution can thus be traced, even to the final equation, in advance of the actual work of combining the equations. This drill is especially helpful in giving the student a clear understanding of the method of procedure.

5. In combining equations in the process of elimination the student should be especially careful that each unknown quantity appears in only one term of the resulting equation. He should also be careful, when occasion offers, to cancel common factors in equations used.

6. Practice and experience will enable the student to avail himself of many devices for shortening the solution of particular examples. He should always be on the alert to discover and make use of any such artifice. The following are examples of devices sometimes employed:

(a) Selecting as an eliminating equation, one obtained by adding the equations given.

(b) Treating complex expressions involving unknown quantities as single quantities and deriving their values. A new group of much simpler equations is thus obtained.

(c) Selecting one unknown quantity and omitting one equation, combine the other equations in such manner as to express the value of each of the remaining unknown quantities, in terms of the one selected; then substitute these values in the omitted equation.

165. Illustrative Examples.

1. Solve the equations
$$\left\{\begin{array}{ll}(1) & x+3y+2z=-1,\\ (2) & 3x+y-z=19,\\ (3) & 2x-2y+3z=8.\end{array}\right.$$

We choose (2) as the eliminating equation.

Multiplying (2) by 3, (4) $9x+3y-3z=57,$

(1) $\underline{x+3y+2z=-1.}$

Subtracting, (5) $8x-5z=58.$

Multiplying (2) by 2, (6) $6x+2y-2z=38.$

(3) $\underline{2x-2y+3z=8.}$

Adding (6) and (3), (7) $8x+z=46.$

(5) $8x-5z=58.$

Subtracting (8) $6z=-12.$

(9) $z=-2.$

Substituting in 7, (10) $8x-2=46.$

(11) $8x=48.$

(12) $x=6.$

Substituting these values of x and z in (2), (13) $18+y+2=19.$

(14) $y=-1.$

2. Solve the equations
$$\left\{\begin{array}{ll}(1) & x-3y-4z+5u=21,\\ (2) & x-y+2z+u=3,\\ (3) & 2x-y+3z+2u=2,\\ (4) & 4x+6y-3z-5u=-15.\end{array}\right.$$

The best moves in an example like the above can be mentally marked out as follows:

Choose (2) as the eliminating equation, both because it has the simplest coefficients and because between (2) and (3) x and u can be eliminated at one move. We will take x as the quantity eliminated.

Combining (1) and (2) we have (5) containing y, z, and u.

Combining (3) and (2) we have (6) containing y and z.

Combining (4) and (2) we have (7) containing y, z and u.

Combining (5) and (7), eliminating u, we have (8) containing y and z.

Combining (6) and (8) and eliminating either y or z, we have an equation from which we may derive the value of the other unknown quantity.

3. Solve the equations

$$\begin{cases} (1) \quad \dfrac{x}{a+b} + \dfrac{a}{a-b} = \dfrac{y-b}{a+b} + \dfrac{x}{a-b}, \\[2ex] (2) \quad \dfrac{x}{a-b} - \dfrac{a^2-3ab-2b^2}{a^2-b^2} = \dfrac{y+b}{a-b} - \dfrac{x}{a+b}. \end{cases}$$

Clearing of fractions,

$$(3) \quad (a-b)\,x + a^2 + ab = (a-b)\,y - ab + b^2 + (a+b)x.$$

$$(4) \quad (a+b)\,x - a^2 + 3ab + 2b^2 = (a+b)y + ab + b^2 - (a-b)x.$$

Collecting terms,

$$(5) \quad -2bx - (a-b)y = -a^2 - 2ab + b^2.$$

$$(6) \quad 2ax - (a+b)y = a^2 - 2ab - b^2.$$

Multiplying (5) by a,

$$(7) \quad -2abx - (a^2-ab)y = -a^3 - 2a^2b + ab^2.$$

Multiplying (6) by b,

$$(8)\quad 2abx-(ab+b^2)y=a^2b-2ab^2-b^3.$$

Adding (8) to (7),

$$(9)\quad -(a^2+b^2)y=-a^3-a^2b-ab^2-b^3.$$

$$(10)\quad y=a+b.$$

Substituting this value of y in (6),

$$(11)\quad 2ax-a^2-2ab-b^2=a^2-2ab-b^2.$$

$$(12)\quad 2ax=2a^2.$$

$$(13)\quad x=a.$$

$$\text{4. Solve the equations}\begin{cases}(1) & \dfrac{x+y+z}{2}+\dfrac{z-x}{4}=2\dfrac{1}{2},\\[2ex] (2) & \dfrac{z-x}{4}+\dfrac{y-z}{3}=-\dfrac{1}{6},\\[2ex] (3) & \dfrac{y-z}{3}+\dfrac{x+y+z}{2}=3\dfrac{1}{3}.\end{cases}$$

It will be noticed that these equations involve three different expressions. These can be treated as single quantities and eliminated as follows:

Subtracting (2) from (1),

$$(4)\quad \frac{x+y+z}{2}-\frac{y-z}{3}=2\frac{2}{3}.$$

$$(3)\quad \frac{y-z}{3}+\frac{x+y+z}{2}=3\frac{1}{3}.$$

Adding (4) and (3),

$$(5)\quad x+y+z=6.$$

Subtracting (4) from (3),

$$(6)\quad \frac{2(y-z)}{3}=\tfrac{2}{3},\text{ or } y-z=1.$$

Substituting the value from (5) in (1),

$$(7)\quad \frac{z-x}{4}=-\tfrac{1}{2},\text{ or } z-x=-2.$$

(5), (6) and (7) are a simpler set of simultaneous equations than (1), (2) and (3), and may be readily solved by the method suggested in Remark 6, (c) thus:

$$\begin{cases} (5) \qquad x+y+z=6. \\ (6) \qquad y-z=1. \\ (7) \qquad z-x=-2. \end{cases}$$

Selecting the unknown quantity z, and omitting equation (5), we have:

From (6), (8) $y=z+1.$

From (7), (9) $x=z+2.$

Substituting these values in (5),

(10) $z+2+z+1+z=6.$

(11) $3z=3.$

(12) $z=1.$

Substituting this value in (8), (13) $y=2.$

Substituting in (9), (14) $x=3.$

EXAMPLES.

Solve the following equations:

1. $\begin{cases} 5x-2y=1. \\ 3x+7y=17. \end{cases}$

2. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{a_1}=\dfrac{1}{a}. \\ \dfrac{x}{a_1}-\dfrac{y}{a}=\dfrac{1}{a_1}. \end{cases}$

3. $\begin{cases} \dfrac{x-\frac{1}{3}}{y+\frac{3}{2}}=\dfrac{x-\frac{2}{3}}{y+\frac{5}{6}}. \\ \dfrac{3x-1}{3y+\frac{5}{2}}=\dfrac{x+\frac{5}{3}}{y+\frac{7}{2}}. \end{cases}$

4. $$\begin{cases} \dfrac{3y-5x}{4}-\dfrac{2x-3y-3}{5}=\dfrac{y}{2}+11\dfrac{1}{2}. \\ 4\dfrac{1}{3}\left(\dfrac{y}{7}-\dfrac{y+x}{2}\right)-\dfrac{3}{2}\left(\dfrac{2y}{3}-\dfrac{5+5x}{6}\right)=-\dfrac{83}{6}. \end{cases}$$

5. $$\begin{cases} \dfrac{x}{m+n}-\dfrac{y}{m-n}=1+\dfrac{3n}{m-n}. \\ x+y=m. \end{cases}$$

6. $$\begin{cases} 3x+\dfrac{2}{y}=1. \\ 2x+\dfrac{3}{y}=-1. \end{cases}$$

7. $$\begin{cases} \dfrac{x}{a-b}-\dfrac{y}{a+b}=\dfrac{1}{b-a}. \\ \dfrac{x}{a+b}-\dfrac{y}{a-b}=-\dfrac{1}{a+b}. \end{cases}$$

8. $$\begin{cases} \dfrac{x}{a}-\dfrac{y}{b}=c. \\ \dfrac{\dfrac{x}{a}}{\dfrac{y}{b}}=2-b. \end{cases}$$

9. $$\begin{cases} 2ax^{-1}+by^{-1}=\dfrac{2ac}{b}. \\ bx^{-1}+cy^{-1}=\dfrac{b^2}{2a}. \end{cases}$$

10. $$\begin{cases} \dfrac{7}{x-1}-\dfrac{2}{y-2}=6\dfrac{1}{2}. \\ \dfrac{3}{x-1}+\dfrac{4}{y-2}=4. \end{cases}$$

11. $\begin{cases} \dfrac{1}{4x} - \dfrac{1}{5y} = \dfrac{1}{5}. \\[2ex] \dfrac{2x - 3y + 5xy}{5xy} = \dfrac{1}{5}. \end{cases}$

12. $\begin{cases} \dfrac{3}{2x} - \dfrac{4y}{y - 2x + 3} = 2\dfrac{1}{2}. \\[2ex] \dfrac{1}{x} + \dfrac{y}{y - 2x + 3} = -\dfrac{1}{6}. \end{cases}$

13. $\begin{cases} \dfrac{(a+b)\,x}{a-b} - \dfrac{(a-b)\,y}{a+b} = \left(\dfrac{1}{a-b} - \dfrac{1}{a+b}\right) 2b. \\[2ex] (x+y)\,(a^2+b^2) = 2a^2 - 2b^2 + 2ab\,(x+y). \end{cases}$

14. $\begin{cases} \dfrac{x}{m} - \dfrac{y}{n} + \dfrac{x}{p} = \dfrac{1}{mnp}. \\[2ex] \dfrac{y}{m} - \dfrac{x}{n} + \dfrac{y}{p} = \dfrac{1}{mnp}. \end{cases}$

15. $\begin{cases} x - y + 6z = 17. \\ x + 3z = 10. \\ 3x - 2y = -1. \end{cases}$

16. $\begin{cases} x + y + z = 12. \\ x + y + u = 3. \\ x + z + u = 5. \\ y + z + u = 7. \end{cases}$

Suggestion. Add the equations.

17. $\begin{cases} \dfrac{1}{2}(x+y) - \dfrac{1}{3}z = 1. \\[2ex] \dfrac{1}{3}(y+z) - \dfrac{1}{4}x = 1\dfrac{3}{4}. \\[2ex] \dfrac{1}{4}(x+z) - \dfrac{1}{5}y = \dfrac{2}{5}. \end{cases}$

18. $$\begin{cases} x - \dfrac{y - z}{3} = y - \dfrac{z - x}{2}. \\ \dfrac{20\,(y - z)}{3\,(x + 1)} = \dfrac{1}{4}. \\ x + z - 1 - y = \dfrac{43}{7}(x - y - z). \end{cases}$$

19. $$\begin{cases} \dfrac{2a}{x} - \dfrac{a_1}{y} = 1. \\ \dfrac{a}{y} + \dfrac{a_1}{z} = 1 + \dfrac{a}{a_1}. \\ \dfrac{a}{z} - \dfrac{a_1}{x} = \dfrac{a^2 - a_1^2}{aa_1} \end{cases}$$

20. $$\begin{cases} x - y - z = 0. \\ m\,(x - y) - (n + p)\,z = py - nx. \\ px - ny + mz = 4m^2. \end{cases}$$

21. $$\begin{cases} x - ay + a^2z = a^3. \\ x + a_1y + a_1^2z = -a_1^3. \\ x + a_2y + a_2^2z = -a_2^3. \end{cases}$$

22. $$\begin{cases} (x + y) - \dfrac{y - z}{7} = 8\dfrac{6}{7}. \\ \dfrac{x - z}{5} - (x + y) = -8\dfrac{3}{5}. \\ \dfrac{y - z}{7} + \dfrac{x - z}{5} = \dfrac{19}{35}. \end{cases}$$

23. $$\begin{cases} x + y = \frac{3}{2}. \\ x - z = \frac{2}{3}. \\ y + u = \frac{3}{4}. \\ z + u = \frac{7}{12}. \end{cases}$$

24. $$\begin{cases} \frac{1}{x} + \frac{1}{2y} + \frac{1}{3z} = 3. \\ \frac{1}{x} - \frac{1}{y} - \frac{1}{2u} = -3. \\ \frac{1}{2x} - \frac{1}{3y} + \frac{1}{6z} = \frac{1}{3}. \\ \frac{3}{2x} + \frac{2}{u} + \frac{3}{z} = 18\frac{1}{2}. \end{cases}$$

25. $$\begin{cases} u + 2x + 3y + 4z = 20. \\ 2u + x - y - 3z = 6. \\ 2u + x + 6y - z = 22. \\ 3u - 5x - 1\frac{1}{2}y + z = -5. \end{cases}$$

26. $$\left.\begin{cases} a + b + c + d = x, \\ 2a + 3b - 2c - 3d = 2x, \\ 3a - 2c + 3d = 3x, \\ 2a - b - c - d = 4x. \end{cases}\right\}$$; find a, b, c and d, when x is known.

27. $$\begin{cases} x - 2y + 3z = 6. \\ y + 3z - 3u = -1. \\ 3x + y + 2u = 13. \\ -3x - 6z + u = -17. \end{cases}$$

28. $$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{ac}.$$
$$\frac{b}{x} - \frac{c}{y} + \frac{a}{z} = \frac{1}{bc} - \frac{1}{ab} - \frac{1}{ac}.$$
$$\frac{c}{x} + \frac{a}{y} + \frac{b}{z} = \frac{1}{bc} - \frac{1}{ab} + \frac{1}{ac}.$$

166. Problems leading to simultaneous equations. In the solution of problems, it is often a matter of judgment as to the number of basal unknown quantities which it is best to use. A careful study of the problem should be made before the number is decided upon. The conditions given must be sufficient to determine as many independent equations as there are unknown quantities chosen. In the problems which follow, it will be convenient to employ at least two basal unknown quantities, and in some cases three or more.

Illustrative Example.

For \$1 I can buy a pounds of tea and b pounds of coffee, or c pounds of tea and d pounds of coffee. What is the price of each per pound?

Let $x =$ the price of tea per pound,
and $y =$ the price of coffee per pound.
$ax =$ the price of a pounds of tea.
$by =$ the price of b pounds of coffee.
$ax+by =$ the price of a pounds of tea and b pounds of coffee.
\$1 = the price of a pounds of tea and b pounds of coffee.
(1) $ax + by = 1.$
$cx =$ the price of c pounds of tea.
$dy =$ the price of d pounds of coffee.
$cx+dy =$ the price of c pounds of tea and d pounds of coffee.
\$1 = the price of c pounds of tea and d pounds of coffee
(2) $cx + dy = 1.$

Solving equations (1) and (2), $x = \frac{b-d}{bc-ad}$ and

$$y = \frac{c-a}{bc-ad}.$$

PROBLEMS.

4 1. A man has for sale a number of sheep, horses and cows. The sheep being valued at \$4 apiece, he offers 2 cows and a sheep for 4/5 the price of a horse,

or 2 horses and 5 sheep for the same as 6 cows. What is the price he asks for a horse and for a cow?

2. Two men together have \$1500; the one who has the greater sum invests his share at 6%, and the other his share at 7%. At the end of one year and 3 months they have \$1620. How much had each at first?

3. A, B and C have among them a dollars. If A had $\frac{1}{3}$ of B's money he could buy a house worth b dollars. It would then be necessary for C to borrow $\frac{1}{2}$ of what B has remaining in order to buy a team and carriage worth c dollars. How much had each at first?

4. A man has three farms. He sells $\frac{1}{2}$ of the first at \$50 per acre, $\frac{1}{4}$ of the second at \$40, and $\frac{1}{5}$ of the third at \$25, and receives \$14000. He then sells from the second as many acres as there are remaining in the first farm and has 150 acres still in the second. Finally he sells $\frac{1}{7}$ of all he has at \$30 per acre and receives for this $2\frac{1}{2}$ times as many dollars as he had acres in the three farms at first. How many acres had he originally in each farm?

5. If you subtract the tens digit of a certain number from the hundreds digit you have 1 more than the figure in units place; if you add the units digit to the hundreds digit, you have 13 minus the tens digit, and if you divide the number by the sum of the digits, you have 57 as a quotient and 1 as a remainder. What is the number?

6. A man has a horse, a carriage and a harness which he offers to sell at \$300. He is offered a horse of the same value as the one he already has, for the

carriage and harness. He accepts the offer and afterward sells the two horses for six times the difference between the value of the carriage and harness. What was the value of each?

7. A man and a boy are in step, and walking in the same direction at the same rate. They are not in step again until they have together taken 22 steps. If the man's step is $1\frac{1}{5}$ the boy's, how many steps has each taken?

8. A store room has been filled by having packed in it 12 barrels and 5 boxes of uniform size. After 4 barrels and 3 boxes have been removed, it is $\frac{3}{5}$ filled. How many barrels will the room hold? How many boxes?

9. A and B start from a station M to walk to N, B starting 20 minutes after A. At the same time that A starts, C leaves N for M. A walks d miles an hour, B, $d+1$ miles, and C, $d+2$ miles. They all meet at the same instant. How far has each walked?

10. On a passenger train from A to B, 80 miles apart, it was found that 7 of those who traveled first class paid 80 cents more than 8 of those who traveled second class. There were 5 more second than first class passengers, and the fare collected was $76. At C, 20 miles from A, 4 first class and 3 second class passengers boarded the train for B, in consequence of which $11.70 additional fare was collected. What was the number of first, and of second class passengers, and what was the first and second class fare from A to B?

11. Two boys, John and Henry, together hoe a

garden in $7\frac{1}{5}$ days. If they work separately, John working 9 days and Henry 4 days, the work will be completed. How long will it take each separately to hoe the garden, if each hoes $\frac{1}{5}$ more in a day when working alone?

12. I loaned a sum of money at a certain rate per cent for 4 years and 6 months, and three months later I loaned $\frac{1}{3}$ the amount of the first loan at 2% more per annum. Both loans were paid when the first loan expired, at which time principal and interest amounted to $2575. If I had loaned both sums at first at the lower rate, I would have received as interest $540. What was the amount of each loan, and the rate of interest?

13. A grocer has two grades of coffee of which he makes one mixture consisting of 9 pounds of the better to 5 pounds of the poorer, and another mixture 2 pounds of the better to 5 pounds of the poorer. How much must he take from each mixture in order to have thus taken out 5 pounds of each grade of coffee?

14. A tank which holds 120 gallons of water has 3 pipes opening into it. If the first pipe is opened, then 2 minutes later the second, and 2 minutes after that the third and all are left opened until the last has been flowing 6 minutes, the tank will contain 108 gallons. If all are opened at the same time and flow for 3 minutes, there will be 39 gallons in the tank. The first and third together fill the tank in 12 minutes. How long will it take each separately to fill it?

15. If a man had $2.50 in pieces of a certain

denomination he would have m pieces; if in pieces of another denomination he would have n pieces. How many pieces of each of the above denominations must be given him, that he may have $5 in 7 pieces?

16. A man played a certain number of games, betting $4 to $3 on every game. At the end he had $2 more than at the beginning. If he had won one less game, his losses would have exceeded his gains by half as many dollars as the number of games played. How many games did he win? How many did he lose?

17. A dog 160 rods behind a rabbit keeps up a certain speed for 10 minutes, and is then obliged to slacken to $\frac{2}{3}$ its previous rate, but at the same time the rabbit slackens to $\frac{1}{2}$ the rate at which it has been running, so that the dog catches the rabbit in 5 minutes. If they had both kept up the first rate of speed 2 minutes longer and then slackened as before, it would have required 16 minutes for the entire race. What was the first rate of each in miles per hour?

18. A man rowing at a certain rate makes the round trip from A up stream to B, 24 miles distant, and return in 5 hours. Having 6 hours at his disposal, he starts to make the trip, but when 16 miles from A, the boat springs a leak which causes him to land. In so doing he loses 40 minutes, but by walking at $\frac{3}{4}$ the speed the boat would have carried him, he is able to spend an hour in B and reach home at the required time by a train moving at twice the speed which the boat would have moved down stream.

What is the rate in miles per hour of the man rowing in still water and what is the rate of the current?

19. On a double track from M to N which are 8 miles apart, a freight and a passenger train are running. The freight starts 576 rods ahead of the passenger train, but the latter reaches N $3\frac{1}{2}$ minutes before the former. On another day, running at the same rate between these stations, the freight starts 9 minutes before the passenger train and reaches N when the passenger train is $\frac{2}{3}$ of a mile from that station. What is the rate of each train in miles per hour?

CHAPTER XII.

DISCUSSION OF PROBLEMS.

167. Indeterminate Problems. It was shown in article 153 that any single equation containing two or more unknown quantities is indeterminate; that is, an indefinite number of sets of values of the unknown quantities can be found which will satisfy the equation. If the number of unknown quantities in any group of simultaneous simple equations exceeds the number of equations, it is evident that the process of elimination will result in a single equation containing more than one unknown quantity. Thus, four independent equations involving five unknown quantities will lead to one equation containing two unknown quantities; and in general m independent equations involving $m + n$ unknown quantities will give an equation containing $n + 1$ such quantities. Since this final equation is indeterminate, it is clear that an infinite number of sets of values of the unknown quantities can be found which will satisfy the original equations. These equations are therefore indeterminate.

It follows from this discussion, that if the conditions in a problem do not give rise to as many independent equations as there are basal unknown quantities, then such unknown quantities may have an indefinite number of values, and the problem is indeterminate.

Consider the following problem:

A newsboy had three different papers for sale, "Herald," "Times" and "Tribune." He sold half as many copies of the "Herald" as of the "Times" and "Tribune," and as many copies of the "Tribune" as of both the "Times" and "Herald." How many copies had he of each?

Let x = The number of copies of the "Herald."
y = The number of copies of the "Times."
z = The number of copies of the "Tribune."

The conditions of the problem give the following equations:

$$(1)\ x = \frac{y+z}{2}$$

$$(2)\ z = x + y$$

Clearing of fractions (3) $2x - y - z = 0$
(4) $x + y - z = 0$

Subtracting (4) from (3) (5) $x - 2y = 0$ or $x = 2y$

Any value may be assigned to y:
If $y = 1$ then $x = 2$ and $z = 3$.
If $y = 2$ then $x = 4$ and $z = 6$.
If $y = 3$ then $x = 6$ and $z = 9$ etc.

Thus not one set of values alone but an indefinite number will satisfy the equation. Hence the problem is *indeterminate*.

168. Impossible Problems. If the number of unknown quantities involved in a group of equations is less than the number of equations, a set of values for such quantities can be found by simply using as many independent equations as there are unknown quantities. The remaining equations may or may not be satisfied by the values obtained. Those that can be deduced from the equations used, either by reduction or combination, will be satisfied. They will thus be shown to express a relation between the unknown quantities not inconsistent with the relations expressed by the equations employed. Such equations are however unnecessary, and may therefore be called redundant. Those of the omitted equations, which are not satisfied by the derived values, are shown by such failure, to express conditions in conflict with those expressed by the equations used in obtaining these values.

If the conditions of a problem when expressed algebraically give a greater number of equations than are necessary to determine the set of values for the unknown quantities involved, the excess equations, as we have seen, are redundant or inconsistent. Redundant equations represent simply unnecessary conditions in the problem; inconsistent equations, conditions which are in conflict with others given, and which, therefore, cannot be fulfilled. In the latter case the problem is impossible.

Consider the following problem:

Find two numbers whose sum is 25 and whose difference is 5, such that if you divide the greater by the less you have 2 as a quotient and 1 as a remainder.

The conditions of this problem lead to the following equations:

$$(1)\qquad x + y = 25.$$

$$(2)\qquad x - y = 5.$$

$$(3)\qquad \frac{x}{y} = 2 + \frac{1}{y}.$$

Here we have three equations containing two unknown quantities, x and y. Values for these quantities derived by combining any two equations will not satisfy the third, as for example:

Solving (1) and (2), we get $x = 15$ and $y = 10$. These values when substituted in equation (3) will not satisfy it.

This shows that only two of the three conditions expressed in the problem can be fullfilled. The solution of the problem is therefore *impossible*.

169. Special and General Problems. Formulas.

If the known quantities in a problem are numerical the answer will be a numerical quantity. Such problems are *special problems*.

If any known quantities are literal the answer will be a literal expression. The problem is then, as regards these quantities, a *general problem*.

If all the known quantities are literal the result obtained is called a *formula*. By substituting for the quantitative symbols in the formula their values, occurring in any special problem of this class, the answer to such problem can be obtained.

Consider the problem:

What number must be added to a in order that if this sum is divided by b the quotient will be c with d as a remainder?

Let $x =$ the required number.

The conditions of the problem give rise to the following equation:

$$\frac{x+a}{b} = c + \frac{d}{b}$$

Solving this equation, we have $x = bc + d - a$.

This answer is a formula which can be employed to find the answer to any special problem of this class.

Take the special problem:

What number must be added to 5 in order that if this sum is divided by 4, the quotient will be 2 with 3 as a remainder?

In this problem $a=5$, $b=4$, $c=2$ and $d=3$. Substituting these values in the formula

$$x=bc+d-a,$$
$$x=4\times 2+3-5$$
$$x=6.$$

170. Negative Results. The values of the known quantities in a problem may be such as to give a negative result.

Take the special problem:

What number must be added to 13 in order that if this sum is divided by 3, the quotient will be 1 with 2 as a remainder?

In this problem $a=13$, $b=3$, $c=1$ and $d=2$. Substituting, as before, in the formula

$$x=bc+d-a,$$
$$x=3\times 1+2-13$$
$$x=-8.$$

How is this answer, -8, to be interpreted?

In interpreting this result it must be borne in mind that while in arithmetic only positive quantities are dealt with, in algebra both positive and negative quantities are considered. Both classes are subject to the same operations, such as addition, subtraction, etc. In the special problem considered, the negative quantity -8 when algebraically added to 13 gives as a result 5 which divided by 3 gives the quotient 1 with 2 as a remainder. This algebraic quantity then meets fully the conditions of the problem.

Since arithmetical quantities are always positive and no positive quantity can be found which will fulfill the conditions of this problem, it follows that this problem is an impossible problem, when considered in an arithmetical sense. The negative sign indicates this.

Noting that the algebraic *addition* of a negative quantity is equivalent to the arithmetical *subtraction* of a positive quantity of the same absolute value, it is evident that the problem can be readily changed so as to be true in an arithmetical sense. It will then read:

What number must be *subtracted* from 13 in order that if this difference is divided by 3, the quotient will be 1 with 2 as a remainder?

Consider the problem:

Two trains are running, one from B east to A at the rate of 3 miles an hour, the other from A west to B at the rate of 4 miles an hour. How far east of C, a town half way between them, will they meet, if A and B are 28 miles apart?

Let $x =$ the distance east of C at which they meet.

$14 + x =$ the distance east of B at which they meet.

$14 - x =$ the distance west of A at which they meet.

$\frac{14 + x}{3} =$ number of hours the east-bound train travels.

$\frac{14 - x}{4} =$ number of hours the west-bound train travels.

$$\frac{14+x}{3}=\frac{14-x}{4}$$

From which $x=-2$.

The negative result shows, in this example as in the other, that the problem, considered in an arithmetical sense, is impossible. It contemplates a positive answer representing distance *east* of C. From the opposite nature of positive and negative quantities, the negative result must indicate distance *west* of C. Algebraically interpreted this answer satisfies all the conditions of the problem.

Since such interpretation shows us that the unknown quantity in the problem is distance *west* instead of *east* of C, we have simply to change *east* into *west* in the problem to make it true in an arithmetical sense.

Hence we conclude:

1. A negative result indicates that the problem as stated is impossible in an arithmetical sense, but is capable of algebraic interpretation.

2. If the statement of the problem be so modified as to make the unknown quantity involved of directly opposite nature, then the problem will be true in an arithmetical sense.

171. Interpretation of $\frac{a}{0}$, $\frac{0}{a}$, $\frac{0}{0}$, $\frac{a}{\infty}$ and Similar Forms.

Algebraic quantities are often divided into two classes, *constants* and *variables*.

172. A constant quantity is one whose value remains the same until changed by the will of the operator.

173. A variable quantity is one whose value is changing. When such change ceases the quantity becomes a constant.

Constant quantities are generally represented by the first, and variable quantities by the last letters of the alphabet.

174. We can conceive a variable quantity to be so increased as to become greater than any assignable quantity. Such quantity is called an **infinite quantity**, an **infinite**, or **infinity**. It is represented by the symbol ∞.

This infinite can be increased until a second infinite is obtained whose value will be infinitely great as compared with that of the first; each resulting order of infinites can be dealt with in a similar manner, and since the mind refuses to entertain any limit to the possible increase in a variable, infinites of a first, second, third and higher orders can be conceived. The relation of these orders will be such that an infinite of any order multiplied by an infinite of the first order will give an infinite of the next higher order.

Let x = an infinite of the first order,
then will x^2 = an infinite of the second order,
x^3 = an infinite of the third order, and so on.

175. An infinitesimal is a quantity which is less than any assignable quantity.

Let us consider the equation $\frac{a}{x} = y$, in which a is regarded as a constant, x as a variable, and y as representing the varying value of the fraction. As x increases in value, y will decrease until, when x becomes greater than any assignable value, y will become less than any assignable value. This infinitely small quantity is called an **infinitesimal**, and is represented by the symbol 0.

This infinitesimal will continue to decrease in the same proportion as the infinite denominator of the fraction increases. Infinitesimals of the first, second, third and higher orders are thus developed which bear to each other the same relation as exists between the corresponding orders of infinites.

Representing infinites of the several orders as before by x, x^2, x^3, etc., then will

$\frac{a}{x} = 0$, an infinitesimal of the first order,

$\frac{a}{x^2} = 0$, an infinitesimal of the second order,

$\frac{a}{x^3} = 0$, an infinitesimal of the third order, and so on.

Quantities can thus be separated into orders ranging from the higher order of infinitesimals to finite quantities and thence on through the successive orders of infinites. These orders are distinct in that you cannot pass from one to the next higher except by an infinite multiplier and hence quantities in the one order have no appreciable value as compared with those in the higher order. Infinitesimals of the first order, for instance, possess no value compared with finite quantities while finite quantities are valueless when compared with the first order of infinites.

175. Speaking then from the standpoint of finite quantities the following important principles may be enunciated:

Principle I. *A finite quantity is not increased or diminished by the addition or subtraction of an infinitesimal.*

Thus, $4 + 0$ (an infinitesimal) $= 4$.

Principle II. *The remainder obtained by subtracting a finite quantity from itself is, in general, an infinitesimal.*

Thus, $4 - 4 = 0$ (an infinitesimal).

176. Definition of Zero. Zero represents an infinitely small quantity or an infinitesimal. It may be obtained by division or subtraction. In either case it represents absence of value as compared with finite quantities.

177. Let us consider the value of $\frac{a}{0}$.

Take $\frac{a}{x} = y$,

in which a is a constant and x and y variables. If x decreases, y will increase. If x continues to decrease, y will continue to increase, until when x becomes infinitely small or an infinitesimal, y will become infinitely large or an infinite.

This may be written,

$$\frac{a}{0} = \infty,$$

and read a divided by zero equals an infinitely large quantity or an infinite.

178. Let us consider the value of $\frac{0}{a}$,

Take $\frac{x}{a} = y$,

in which as before, a is a constant and x and y variables. As x decreases, the value of the fraction represented by y will decrease until, when x becomes an infinitesimal, y will also become infinitely small.

The equation can then be written,

$$\frac{0}{a}=0,$$

and read zero divided by a finite quantity equals zero.

179. Let us consider the value of $\frac{0}{0}$.

Take $\frac{x}{y}=a,$

in which x and y are variables and a represents any finite quantity. If x and y be each divided by 2, the value of the fraction will still equal a. Continuing to divide both numerator and denominator by the same number an infinite number of times, the terms of the fraction will each reduce to an infinitesimal while the value of the fraction will remain unchanged.

The equation may then be written $\frac{0}{0}=a$, and read zero divided by zero equals any finite quantity.

The expression $\frac{0}{0}$ is thus shown to be indeterminate. In certain cases, it will be found that this indeterminateness can be made to disappear.

EXAMPLES.

1. Determine the value of $\frac{a}{\infty}$.

2. Determine the value of $\frac{\infty}{a}$.

3. Determine the value of $\frac{\infty}{\infty}$.

4. Determine the value of $\frac{0}{\infty}$.

5. Determine the value of $\frac{\infty}{0}$.

180. Problem for discussion. Two men travel in the same direction from two places a miles apart; the one at the rate of m miles an hour and the other at the rate of n miles an hour. When will they be together?

Let $x =$ the number of hours before they will be together.

$nx =$ the number of miles traveled by the one pursued.

$mx =$ the number of miles traveled by the one pursuing.

$$mx - nx = a$$

$$x = \frac{a}{m - n} \text{ number of hours.}$$

In this answer a, m and n, are arbitrary constants to which any values may be assigned.

1. By assigning $a = 0$, the answer will reduce to the form $\frac{0}{m - n}$, or zero divided by a constant. We have learned that the value of a fraction of this form is zero. A careful reading of the problem will reveal that the answer is a correct one.

2. By making a a finite quantity and $m = n$, the answer will reduce to $\frac{a}{0}$, or a finite quantity divided by zero. The value of this fraction, as we have found, is an infinite; the answer can readily be interpreted in connection with the problem.

3. If $a = 0$ and $m = n$, the answer will reduce to the form of $\frac{0}{0}$. This form, we have learned, represents in general any value, and such value will meet all the conditions imposed upon the problem.

The student is asked to solve and discuss the following problems:

1. A grocer has two kinds of tea, one worth a cents per pound, the other b cents. How many pounds of each must be taken to form a mixture of n pounds, which shall be worth c cents per pound?

2. The fore-wheel of a carriage is m feet, the hind wheel n feet in circumference. What is the distance passed over when the fore-wheel has made p revolutions more than the hind-wheel?

CHAPTER XIII.

INEQUALITIES.

181. **An inequality** is the symbolic statement that one quantity is algebraically greater or less than another.

182. The **inequality sign** is an angle with the opening toward the greater quantity.

Thus,

$7 > 5$, read seven is greater than five.
$a < b$, read a is less than b.
$5 > 0$, read five is greater than zero.
$-7 < -2$, read -7 is less than -2.

183. The **members** of an inequality are the quantities connected by the inequality sign. The **first member** is on the left and the **second member** on the right of the sign of inequality.

184. Inequalities **subsist in the same sense** when the greater quantity in each is to the left or to the right of the sign of inequality.

Two inequalities **subsist in a contrary sense** when in one the greater quantity is to the right and in the other to the left of the sign of inequality.

185. **Properties of inequalities.** The following properties of inequalities are readily deduced from the definitions given. The student is asked to give and demonstrate them.

1. An inequality will continue to subsist in the same sense:

(a) After the same positive quantity has been added to or subtracted from each member.

(b) After each member has been multiplied or divided by the same positive quantity.

(c) After the addition of an inequality subsisting in the same sense.

(d) After multiplication by an inequality subsisting in the same sense, provided the members of both inequalities are positive quantities.

2. An inequality will subsist in a contrary sense after each member has been multiplied or divided by the same negative quantity.

The following are easily developed from the foregoing properties:

Corollary I. Any term may be transposed from one member of an inequality to the other by changing its sign. (From Property I (a)).

Corollary II. The signs of all the terms of an inequality may be changed by reversing the sign of inequality. (From Property 2.)

There are other properties of inequalities, relating to powers and roots of the members, which can be readily derived by the student after involution and evolution have been considered.

186. It is important in handling inequalities to bear in mind that the square of a quantity, either positive or negative, is greater than zero. Thus, x^2, $(a-b)^2$, $x^2-2xy+y^2$ are each positive quantities and thus greater than zero.

187. Suggestions.

The ordinary operations connected with inequalities may be classified as follows:

(a) Those involving no change in sign of inequality.
    (1) Transposition of terms.
    (2) Clearing of fractions when multiplier is positive.
    (3) Cancellation of common positive factor from the two members.
    (4) Addition of inequalities subsisting in the same sense.
    (5) Multiplication of positive inequalities subsisting in the same sense.
(b) Those requiring a reversal of the sign of inequality.
    (1) Changing signs of all the terms.
    (2) Clearing of fractions when multiplier is negative.
    (3) Cancellation of common negative factor from each member.
(c) Those in which the change in the sign of inequality is dependent upon the relative value of the quantities dealt with.
    (1) Subtracting inequalities which subsist in the same sense.
    (2) Dividing inequalities which subsist in the same sense.

Illustrative Examples.

1. Find the limit of x in $2x-\frac{3}{5}>\frac{12x-1}{7}+\frac{4}{35}$.

Multiplying by 35 $70x-21>60x-5+4$.
Transposing and uniting terms $10x>20$.
Dividing by 10 $x>2$.

2. If x and y are different numbers,

$$\text{is } x^2+\frac{y^2}{4} > \text{or} = \text{or} < xy?$$

Assume $$x^2+\frac{y^2}{4} > \text{or} = \text{or} < xy.$$

Multiplying by 4 $$4x^2+y^2 > \text{or} = \text{or} < 4xy.$$

Transposing $4xy$ $$4x^2-4xy+y^2 > \text{or} = \text{or} < 0.$$

Since every square is positive, we know

$$4x^2-4xy+y^2 \text{ or } (2x-y)^2 > 0.$$

Since the operations performed have not changed the sense of the inequality, therefore

$$x^2+\frac{y^2}{4} > xy.$$

EXAMPLES.

1. Find the limit of x in $\frac{2x-5}{3} > \frac{8+7x}{2}$.

2. What integral values of x will satisfy the two inequalities, $4x+3 > 2x+7$ and $5x-4 < 31-2x$?

3. Find the limits of x and y in

$$4-\frac{2x-y}{3} < x-\frac{7-2y}{4} \text{ and } 2y=3x.$$

4. If x, y, z and u are positive quantities, and $x > y$ and $z > u$, determine whether

$$xz > \text{or} = \text{or} < yu,$$
$$-xz > \text{or} = \text{or} < -yu,$$
$$\frac{x}{z} > \text{or} = \text{or} < \frac{y}{u},$$
$$-\frac{x}{z} > \text{or} = \text{or} < -\frac{y}{u}.$$

5. Between what values must x lie in order that $x+\frac{cn-nx}{m}>c$ and $6x-6b<5ax-5ab$, if $m>n$ and $a<\frac{6}{5}$, m and n being positive?

6. What would be the limits in the above problem if $m<n$ and $a>\frac{6}{5}$?

7. For what value of x is $\frac{3x}{2}-\frac{7x}{4}>\frac{x}{3}-14$?

8. If $mn>21$ and $m<3$ and is positive, show that $n>7$.

9. Show whether the sum of a positive quantity and its reciprocal is $>$ or $=$ or <2.

10. Prove that the sum of the squares of two quantities is greater than twice their product.

11. Show that $\frac{a^2}{y}>2a-y$, y being positive and a not equal to y.

12. Show that $9x^2+12x+11$ cannot be less than 7.

13. Which is the greater $mn^{-2}+m^{-2}n$ or $m^{-1}+n^{-1}$, when m and n are both positive and different?

14. Which is greater $\frac{x^3+y^3}{xy}$ or $x+y$, when x and y are positive?

15. Arrange according to value, $\frac{a+b}{2}$, $\sqrt{ab}$ and $\frac{2ab}{a+b}$, when a and b are positive and different.

16. Determine whether $\frac{x^2y^2+m^2z^2}{2}$ is $>$ or $=$ or $< xy+mz-1$.

17. If $m^2+n^2=2$ and $y^2+z^2=6$, prove that $my+nz<4$.

18. If $x^2+y^2+z^2=1$, when x, y and z are different quantities, show whether $xy+xz+yz>$ or $=$ or <1.

19. Show that if $\frac{a}{b}<1$, then $\frac{a+1}{b+1}>\frac{a}{b}$, a and b being positive.

20. The distance between two towns, A and B, can be expressed as an integral number of miles. A man drives from A to B at the rate of 6 miles an hour and is more than 4 hours in making the journey. He returns at the rate of 5 miles an hour, stops 48 minutes on the way, and is less than 6 hours in completing the journey from B to A. How far apart are B and A?

21. A farmer has $\frac{1}{2}$ as many horses as sheep. If he had 1 more sheep and 8 more horses, he would have more horses than sheep. After he sold 3 sheep and bought two more horses, he had more sheep than horses. How many had he of each at first?

CHAPTER XIV.

INVOLUTION.

188. Involution is the process of raising a quantity to any power.

189. The power of a quantity is the product obtained by taking that quantity any number of times as a factor.

190. The exponent of the power is a number or letter written to the right and a little above the quantity.

191. In determining the meaning of an exponent, three cases arise:

1. When the exponent is a positive integer.
2. When the exponent is a negative integer.
3. When the exponent is a positive or negative fraction.

Only the first two cases will be discussed under involution, the third will be treated under evolution.

Case I. *When the exponent is a positive integer.*

It is clear in this case that the exponent simply indicates the number of times the quantity is taken as a factor.

Thus,

In a^5, a is taken 5 times as a factor.

In a^n, a is taken n times as a factor.

In $(a+b)^r$, $a+b$ is taken r times as a factor.

In $(a^2)^6$, a^2 is taken 6 times as a factor.

In $[(x+y)^2]^m$, $(x+y)^2$ is taken m times as a factor.

Case II. *When the exponent is a negative integer.*

It is evident from the principle of negative exponents that the exponent in this case indicates the reciprocal of what it would signify if positive.

Thus,

$$a^{-3} = \frac{1}{a^3}, \text{ or the reciprocal of } a^3.$$

$$(a^2)^{-n} = \frac{1}{(a^2)^n}, \text{ or the reciprocal of } (a^2)^n.$$

$$(-3)^{-2} = \frac{1}{(-3)^2}, \text{ or the reciprocal of } (-3)^2.$$

In involution the following laws are important:

192. I. Law of Signs. *Even powers of a quantity are positive, while odd powers have the same sign as the quantity itself.*

This follows from the fact that the quantity is repeated as a factor an even or an odd number of times as indicated by the exponent of the power.

193. II. Law of Factors. *In raising a quantity, composed of factors, to any power, each factor must be raised to the required power.*

In obtaining the power by the process of multiplication, each factor of which the quantity is composed will be repeated as many times as the quantity itself is used as a factor, and will consequently be raised to the indicated power.

Thus, $(ab)^2 = ab \times ab = a^2b^2$.

$(a^{-2}(x+y)^2)^3 = a^{-2}(x+y)^2 \times a^{-2}(x+y)^2 \times a^{-2}(x+y)^2 = a^{-6}(x+y)^6$

$(2cd)^n = 2cd \times 2cd \times 2cd \dots$ to n factors $= 2^nc^nd^n$.

194. Any integral power of a factor consisting of a base with any given exponent, may be found by the following rules:

FIRST RULE.

Multiply the exponent of the base by the exponent of the power and attach the result as an exponent to the given base.

It is plain that this rule simply shortens the work of obtaining the power by actual multiplication.

Thus,

$(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$.

$(a^3)^n = a^3 \times a^3 \times a^3 \dots$ to n factors $= a^{3n}$.

$[(xy)^2]^4 = (xy)^2 \times (xy)^2 \times (xy)^2 \times (xy)^2 = (xy)^8$.

$[(a+b)^2]^s = (a+b)^2 \times (a+b)^2 \dots$ to s factors $= (a+b)^{2s}$.

$$[x^2]^{-3} = \frac{1}{(x^2)^3} = \frac{1}{x^2} \times \frac{1}{x^2} \times \frac{1}{x^2} = \frac{1}{x^6} = x^{-6}.$$

$$[(a+b)^2]^{-n} = \frac{1}{(a+b)^2} \times \frac{1}{(a+b)^2} \dots \text{ to } n \text{ factors} = \frac{1}{(a+b)^{2n}} = (a+b)^{-2n}.$$

Find the powers indicated in the following:

$(a^4)^2$
$(a^2)^n$
$(a^3)^{2n}$
$(b^2)^{-3}$
$(b^3)^{-n}$
$(b^{-2})^{-4}$
$(b^n)^{-n}$
$(b^{-2n})^{-3n}$
$[(xy)^{-2}]^3$
$[(ab)^{-3}]^{-n}$
$[(x+y)^n]^{-3}$
$[(x+y)^{-n}]^{-n}$

SECOND RULE.

Raise the base to the required power and attach to the result as its exponent the exponent of the original base.

First Proof.

By the second rule in multiplication, factors whose bases have like exponents are multiplied, by multiplying the bases and attaching to the product the common exponent. Employing this rule, the result will be the power of the base, with an exponent the same as that of the original base.

Second Proof.

$(a^n)^m = a^{nm}$ by first rule.

$(a^m)^n = a^{nm}$ by the first rule.

$\therefore (a^n)^m = (a^m)^n$, hence the rule.

Illustrative Examples.

The third power of $2^1 = 8^1$.

The second power of $2^3 = 4^3$.

The second power of $(a + b)^3 = (a^2 + 2ab + b^2)^3$.

The third power of $3^{-2} = 27^{-2}$.

The second power of $(ab)^n = (a^2 b^2)^n$.

The nth power of $(xy^{-1})^m = (x^n y^{-n})^m$.

The foregoing rules are the *only ones* by which factors or quantities can be raised to a power. Convenience should determine the one to be used. The law of signs and the law of factors are, of course, always to be remembered, and employed in connection with the application of either of the rules.

EXAMPLES.

Find the value of the following:

1. $(3a^2 b^{-4})^2$.
2. $(-7a^2 b^3 c^{-4} x)^3$.
3. $(3ab^2 x^{-1} y^{-2})^{-2}$.
4. $(-4a^2 x^{-3} y^2)^{-2}$.
5. $(-6ab^{-5} x^{-2})^{-3}$.
6. $(a^3 x^{m+2} y^{-P-1})^3$.
7. $(m^i n^{-J})^{-i}$.
8. $(-2x^2 y^{-3} z^4 u^{-1})^{-4}$.
9. $(a^{-2m} b^{-m})^m$.
10. $(3a^n b^m c^2)^0$.
11. $(x^{-m^2})^{m^{-3}}$.
12. $(-11bxy^{-2})^n$.
13. $(5(x+y)^2)^2$.
14. $(-3(x-y)^m (x+a)^{-n})^{-2}$.
15. $(-5a^{-2} c^{-n} (x+y)^s)^{-3}$.
16. $(-2x^{-1} y^{-2} \{z-[a-c]^3\})^{-2}$.

From the rule for the multiplication of fractions it can readily be seen that to raise a fraction to any power, both numerator and denominator must be raised to the required power.

EXAMPLES.

Find the value of the following expressions:

1. $\left(\dfrac{a^2}{b^2}\right)^3$.
2. $\left(-\dfrac{a^3}{bc^2}\right)^5$.

In the following free the results from negative exponents:

3. $\left(\dfrac{a^{-2} b^2}{cd^{-m}}\right)^4$.
4. $\left(-\dfrac{xy^{-3}}{mn^{-4}}\right)^3$.
5. $\left(-\dfrac{2a^{-m} x^2}{bc^{-n} y^2}\right)^{-4}$.
6. $\left(\dfrac{2a^{-2}(x+y)^3}{3b(x-y)^2}\right)^2$.

7. $\left(-\frac{3xy^{-2}(a-b)^{-3}}{5z^{-4}\,(a+b)^2}\right)^{-2}$

8. $\left(-\frac{ax^2\,(x-a)}{3by^{-2}(a-b)^{-2}}\right)^m$.

195. Powers of Polynomials. Any polynomial may be regarded as a quantity consisting of a base with an exponent 1. Its power is the power of the base with the same exponent, 1. It thus fits under the *second rule*. The required power of the base may be found by continuous multiplication.

EXAMPLES.

1. Find the square of $2a+3b^2$.

2. Find the square of $a+m-n$.

3. Find the cube of $a+\frac{1}{2}$.

4. Find the cube of $2a^2+3a-1$.

5. Find the square of $\frac{ax-by}{ay-bx}$.

By noticing with care the process of multiplication, the steps taken in certain cases may be stated in the form of rules of considerable practical utility.

The following are illustrations:

Square of a Polynomial. The square of a polynomial is equal to the square of each term, together with the sum of the products of twice each term into each of the succeeding terms.

Cube of a Polynomial. The cube of a polynomial is equal to the sum of the cubes of its terms, plus three times the product of the square of each term by each of the other terms, plus six times all the possible products which can be formed by multiplying together any three of its terms.

EXAMPLES.

1. Find the square of $a - b + c$.
2. Find the square of $a + b - c + d - f$.
3. Find the square of $1 + 2x + 3x^2$.
4. Find the square of $1 - x + x^2 - x^3$.
5. Find the square of $a - 2b + 3ab - m$.
6. Find the square of $3x^3 - 2x^2 - x + 4$.
7. Find the square of $x^3 + 5x^2 - x + 2$.
8. Find the cube of $a + b + c$.
9. Find the cube of $a - b + c - d$.
10. Find the cube of $2x^2 - 3x - 1$.
11. Find the cube of $1 + x + x^2 + x^3$.
12. Find the cube of $2x^3 - x^2 + 2x - 3$.

CHAPTER XV.

EVOLUTION.

196. Evolution is the process of finding any root of a quantity.

197. The root of a quantity is one of the equal factors into which the quantity is conceived to be resolved.

If we conceive the quantity to be resolved into two equal factors, one of the factors is the square root; if resolved into three equal factors, one of the factors is the cube root; if resolved into n equal factors, one of the factors is the nth root.

The root is indicated by the radical sign, $\sqrt{\ }$

$\sqrt{\ }$ denotes the square root.

$\sqrt[3]{\ }$ denotes the cube root.

$\sqrt[n]{\ }$ denotes the nth root.

198. The letter or figure to the left and a little above the radical is called the **index** of the root.

199. If the root can be exactly extracted the quantity is called a **rational quantity**.

$\sqrt{4}$ and $\sqrt[3]{8}$ are rational quantities.

200. If the root cannot be exactly extracted, the quantity is an **irrational** or **surd quantity**.

$\sqrt{3}$ and $\sqrt[3]{9}$ are irrational quantities.

201. A second method of indicating the root of a quantity is by means of a **fractional exponent.**

$a^{\frac{1}{2}}$ is one of the two equal factors of a, $\therefore a^{\frac{1}{2}}$ is the square root of a, or algebraically expressed $a^{\frac{1}{2}} = \sqrt{a}$.

$a^{\frac{1}{3}}$ is one of the three equal factors of a $\therefore a^{\frac{1}{3}} = \sqrt[3]{a}$.

$a^{\frac{1}{n}}$ is one of the n equal factors of a $\therefore a^{\frac{1}{n}} = \sqrt[n]{a}$.

$a^{\frac{m}{n}}$ is one of the n equal factors of a^m $\therefore a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

We know now that $a^{\frac{m}{n}}$ equals the nth root of the mth power of a. $a^{\frac{m}{n}}$ also equals the mth power of the nth root of a, $\therefore$ the nth root of the mth power equals the mth power of the nth root; that is, in a fractional exponent, the numerator denotes a power and the denominator a root, and the value of the expression is the same no matter in which order these operations are conceived as performed.

Thus we reach an interpretation of that case necessarily omitted in involution, viz., the meaning of that *power of a quantity denoted by a fractional exponent.*

We can now say:

$a^{\frac{m}{n}}$ means that one of the n equal factors into which we conceived a to be resolved is taken m times, or $a^{\frac{m}{n}}$ is one of the n equal factors into which a raised to the mth power is resolved.

202. We are now prepared, as a preliminary step to deriving *rules of evolution*, to prove that the two *rules of involution* hold true in this omitted case, and are therefore general in their application.

Proof for Rule 1.

$(a^p)^{\frac{m}{n}}$ is the nth root of $(a^p)^m$; but $(a^p)^m = a^{pm}$, hence the nth root of $(a^p)^m$ becomes the nth root of a^{pm}, which may be written $a^{\frac{pm}{n}}$ and therefore $(a^p)^{\frac{m}{n}} = a^{\frac{pm}{n}}$, and it is proven that the *first rule* covers the case.

Proof for Rule II.

By the above $(a^p)^{\frac{m}{n}} = a^{\frac{pm}{n}}$.

By the *first rule* $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{pm}{n}}$.

$\therefore (a^p)^{\frac{m}{n}} = \left(a^{\frac{m}{n}}\right)^p$ and the *second rule* covers the case.

The rules of involution are thus proven general.

203. *Evolution is the converse of involution.* In involution the equal factors of roots are given to find the product or power; in evolution, the product or power is given, to find the equal factors or root.

The *power* and the *root* are correlative terms; 8 is the third power of 2, and 2 is the third root of 8.

The laws and rules of evolution must then be the converse of those of involution.

204. In involution the even powers of both positive and negative quantities are positive while the odd powers have the same sign as the quantities themselves. Since in evolution the converse is true, we have the following as the

Law of Signs. *Even roots of any positive quantity may be either positive or negative, while odd roots will have the same sign as the quantity.*

205. In raising a quantity to any power, each of its factors is raised to the required power. The converse of this gives us in evolution the following as the

Law of Factors. *In extracting the root of a quantity composed of factors, the required root of each factor must be extracted.*

206. Taking the converse of the two rules in involution relating to factors, we have as the two rules for extracting the root of any factor the following:

FIRST RULE.

Divide the exponent of the base by the index of the required root and attach the result as an exponent to the given base.

SECOND RULE.

Extract the required root of the base and attach to the result as its exponent the exponent of the original base.

These two rules taken in connection with the law of signs and factors, will solve all examples. Care should be taken in the choice of rules in each example.

EXAMPLES.

1. Find the square root of $64a^6b^4$.
2. Find the square root of $196a^2b^4c^6x^8$.
3. Find the cube root of $125a^3b^6$.
4. Find the cube root of $-216x^6y^9$.
5. Find the fourth root of $16a^4b^8c^{12}$.

6. Find the fifth root of $-32a^{15}(x+y)^{10}$.

7. Find the nth root of $3^n a^{2n} b^{3n}$.

8. Find the cube root of $x^{3n+3} y^{9m+6}$.

9. Find the square root of $\dfrac{25a^4c^4x^6}{64m^4n^8}$.

10. Find the square root of $\dfrac{9a^{-2}b^4}{16x^6m^{12}}$.

11. Find the cube root of $\dfrac{27a^3c^6}{8m^3x^9}$.

12. Find the cube root of $-\dfrac{125a^3b^9x^{12}}{216c^6y^9}$.

13. Find the square root of $64a^{-4}b^2x^6$.

14. Find the cube root of $-512a^3b^{-9}c^{-6}$.

15. Find the fourth root of $256x^{-4}y^{-8}z^{-12}$.

16. Find the fifth root of $-32a^{-10}b^{-5}c^{15}$.

17. Find the square root of $64a^{-2m}b^{-4n}$.

18. Find the nth root of $a^{-mn}y^n$.

19. Find the mth root of $x^{-mn}y^{m^2}z^{-m^3}$.

20. Find the nth root of $a^2b^{-1}c^{-1}$.

21. Find the square root of $(a^2+2ab+b^2)^3$.

22. Find the square root of $(9x^2-12xy+4y^2)^n$.

23. Find the square root of

$$16a^{-4}b^2(25m^2+10mn+n^2)^5.$$

24. Find the square root of

$$36a^{-6}(a+b)^4(a^2-2ab+b^2)^{-8}.$$

25. Find the square root of $\dfrac{81a^{-10}b^{-6}(x-y)^4}{49c^2(16x^2+4x+\frac{1}{4})^7}$

207. Roots of Polynomials.

Any root of a polynomial may be considered as coming under the *second rule*, the polynomial being regarded as a quantity consisting of a base with the exponent 1. In finding the root of the quantity, the steps to be taken are simply the converse of those by which the power was obtained. It is by noting with care these latter steps and reversing them that we obtain the rule for extracting the root.

208. Square Root of Polynomials.

The square of $a + b$ is $a^2 + 2ab + b^2$, in which a and b represent any quantities. Calling $a + b$ the *root* and $a^2 + 2ab + b^2$ the *power*, we may say that the power equals the square of the first term of the root, plus twice the product of the first term by the second, plus the square of the second term.

Reversing the steps by which the power was obtained from the given root, the root may be obtained from the given power.

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{(Power)}$$

$$\begin{array}{r|l}
 & a^2 + 2ab + b^2\ \underline{(a + b} \quad \text{(Root)} \\
 & a^2 \\ \hline
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}$$

The following are the steps taken, with the reason for each:

Arrange the terms of the power according to the descending exponents of some letter.

Reason. This is simply for convenience. It is the order in which the terms of the power appear when the power is obtained from the root by continuous multiplication.

2. *Extract the square root of the first term of the power and it will give the first term of the root.*

Reason. The square of the first term of the root equals the first term of the power; therefore the square root of the first term of the power equals the first term of the root.

3. *Subtract the square of the root already found from the power.*

Reason. This step is taken in order to obtain the remainder of the power. If the remainder is 0 it shows that the exact root has been found.

4. *Divide the first term of the remainder by two times the first term of the root and it will give the second term of the root.*

Reason. In forming the power, its *second* term, which is the *first* term of the remainder, was obtained by multiplying two times the first term of the root by the second; therefore, if we divide the first term of the remainder by two times the first term of the root, it will give the second term of the root.

5. *Add to twice the first term of the root, the second term of the root, multiply their sum by the second term of the root, and subtract the product from the first remainder.*

Reason. This will give the remainder of the power after there has been taken from it the square of the root already found. The remainder could have been obtained by simply squaring the root and subtracting it from the original power. Since, however, the square of the first term of the root has already been subtracted, the desired result can be obtained by subtracting from the *first remainder*, the remaining part of the square of the root, viz. $2ab+b^2$ or two times the product of the first term of the root by the second, plus the square of the second term of the root. The same result may be reached by subtracting from the *first remainder* $(2a+b)b$ or the product obtained by adding to twice the first term of the root the second term of the root, and multiplying their sum by the second term of the root.

This is the method generally adopted. The *purpose* of this step is to find what remains of the original power after the square of the square root found has been taken from it. The step accomplishes this end, and is therefore a proper and correct step to take.

If the remainder is 0, the root found is the *exact square root* of the power. Attention is called to these various steps in the following example:

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & \multicolumn{1}{l}{9a^2+24ab+16b^2\,\underline{|3a+4b}} \\
\multicolumn{1}{r}{} & \multicolumn{1}{l}{9a^2} \\
\hline
6a+4b & 24ab+16b^2 \\
 & \underline{24ab+16b^2}
\end{array}$$

If there are three monomial terms in the *root*, the power may still be regarded as the square of a binomial; representing the terms of the root by a, b and c, the root may be put in the form of $(a+b)+c$, a binomial consisting of the terms $a+b$ and c. The square of this binomial, under the rule for squaring binomials, will be $(a+b)^2+2(a+b)c+c^2$ or $(a^2+2ab+b^2)+2(a+b)c+c^2$.

By the steps already noted, the binomial term $(a+b)$ can be obtained.

The work, then, will stand as follows:

$$\begin{array}{rl}
 & (a^2+2ab+b^2)+2(a+b)c+c^2 \mid \underline{(a+b).} \\
 & a^2+2ab+b^2 \\ \cline{2-2}
2(a+b) & \!\!\mid 2(a+b)c+c^2.
\end{array}$$

The remainder of the work will be a repetition of steps 4 and 5, simply regarding and speaking of $(a+b)$ as the *first term*.

Extract the square root of

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4.$$

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & a^4-2a^3x+3a^2x^2-2ax^3+x^4 \mid \underline{a^2-ax+x^2} \\
\multicolumn{1}{r}{} & a^4 \\ \cline{2-2}
2a^2-ax & -2a^3x+3a^2x^2-2ax^3+x^4 \\
 & -2a^3x+a^2x^2 \\ \cline{2-2}
2a^2-2ax+x^2 & 2a^2x^2-2ax^3+x^4 \\
 & 2a^2x^2-2ax^3+x^4 \\ \cline{2-2}
\end{array}$$

If there are four monomial terms in the root, represented, say by a, b, c and d, the power can be considered as the square of a binomial, $(a+b+c)+d$, and treated accordingly. The power can thus always be considered as the square of a binomial, no matter how many monomial terms there are in the root.

In certain cases where the square root of a polynomial cannot be exactly obtained, one or more terms of the root may be found. In such cases the square of this root, plus the last remainder, will equal the original quantity.

EXAMPLES.

Find the square root of the following:

1. $4x^4 - 12x^3 + 25x^2 - 24x + 16$.
2. $a^4 - 6a^3x + 13a^2x^2 - 12ax^3 + 4x^4$.
3. $9x^2y^2 + 24x^2y - 30xy^3 + 16x^2 - 40xy^2 + 25y^4$.
4. $6bx^3 - 12bm^2x - 4m^2x^2 + x^4 + 9b^2x^2 + 4m^4$.
5. $10a^4b^2 + a^6 - 4a^5b - 20a^3b^3 + 16b^6 + 25a^2b^4 - 24ab^5$.
6. $a^2 + 2ab + \frac{2a}{c} + b^2 + \frac{2b}{c} + \frac{1}{c^2}$.
7. $x^{2m} + 2x^my^{\frac{n}{2}} + 2x^mz^{\frac{P}{3}} + y^n + 2y^{\frac{n}{2}}z^{\frac{P}{3}} + z^{\frac{2P}{3}}$.
8. $16x - 40x^{\frac{3}{4}} + 49x^{\frac{1}{2}} - 30x^{\frac{1}{4}} + 9$.
9. $\frac{7}{3}a^2 - 4a^3 + 4a^4 - \frac{2}{3}a + \frac{1}{9}$.
10. $4a^{-2} + 12a^{-1}b - 11b^2 - 30ab^3 + 25a^2b^4$.

209. Square Root of Numbers.

Any number may be regarded as an algebraic polynomial composed of units, tens, hundreds, etc. The square of a number is therefore built up in the same way as the square of an algebraic polynomial. Thus,

(tens + units)^2 = the square of the tens, plus twice the product of the tens by the units, plus the square of the units.

$(60 + 3)^2 = 60^2 + 2 \times 60 \times 3 + 3^2 = 3969$.

The terms which make up the square of a number are always blended together, as in the answer to the above example. It is necessary, therefore, in finding the root, to first separate the power into terms. This necessitates the step designated as *separating the number into periods.* The step is explained as follows:

The square of any figure in units' place will contain simply units or units and tens, thus, $3^2 = 9$, $9^2 = 81$.

The square of any figure in tens' place will contain hundreds or hundreds and thousands, thus $(30)^2$ or 3 tens squared $= 900$ or 9 in hundreds' place; $(90)^2$ or 9 tens squared $= 8100$ or 8 in thousands' place and 1 in hundreds' place.

Squaring any figure in tenths, place will give hundredths, or tenths and hundredths, thus, $.2^2 = .04$, $.8^2 = .64$.

In the same way $(.03)^2 = .0009$ and $(.09)^2 = .0081$.

By the formula for squaring a polynomial, it was made clear that the square of a number contains, among other terms, the square of each figure in the number or root; it is now shown that the power having been divided into periods of two figures each, counting in both directions from the decimal point, the square of each figure of the root will be found in a particular period of the power; that is, the square of units will be found in the first period to the left of the decimal point, the square of tens in the second period, and so on; the square of tenths in the first period to the right of the decimal point, the square of hundredths in the second period, and so on.

If the power, then, be separated into periods, as indicated, it will make known the number of figures in the root and their position relative to the decimal point. It will also show us where to look for the square of each root figure.

EXAMPLES IN DIVIDING INTO PERIODS.

1. Divide into periods 243726.

2. Divide into periods 32724.

3. Divide into periods 327.45.

4. Divide into periods 675.4.

5. Divide into periods 42732.567.

6. Put into the form of a decimal and then divide into periods $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$.

Let it now be required to extract the square root of 7569. The process may be illustrated as follows:

$$\begin{array}{r|l|l}
 & \text{tens}^2 + 2\times\text{tens}\times\text{units} + \text{units}^2 & \underline{\text{tens} + \text{units.}} \\
 & \underline{\text{tens}^2} & \\
2\times\text{tens} + \text{units} & 2\times\text{tens}\times\text{units} + \text{units}^2 & \\
 & \underline{2\times\text{tens}\times\text{units} + \text{units}^2.} &
\end{array}$$

$$\begin{array}{r|l|l}
 & & \quad\text{tens}\quad\text{units.} \\
 & 75,69 & \underline{\quad 8 \quad + \quad 7} \\
\text{tens}\quad\text{units} & \underline{64} & \\
2\times 8 + \ 7 & 1169 & \\
 & \underline{1169} &
\end{array}$$

The following are the steps taken with the reason for each step:

1. *Separate the number into periods of two figures each beginning at the decimal point.*

Reason. As already shown, this step divides the power into periods corresponding to the figures of the root, thus determining the number of figures in the root and their position with reference to the decimal point.

In the example given, there are two figures in the root, tens and units.

2. *Extract the square root of the greatest square in the left hand period of the power, and it will give the tens or the first figure of the root.*

Reason. The square of tens or the first figure of the root appears only in the first period of the power; therefore, if we extract the square root of the greatest perfect square in the first period of the power, it will give the first figure of the root.

3. *Substract the square of the first figure of the root from the first period of the power and bring down the next period.*

Reason. This will give the remainder of the power after substracting the square of the root already found.

4. *Divide this remainder by twice the first figure of the root and it will give the second figure of the root.*

Reason. The remainder contains twice the product of the tens by the units, plus the square of the units. These two terms cannot be separated, but as the first is much the greater, it is probable that if we divide the remainder by twice the tens, it will give the units.

5. *To twice the first figure of the root, considered as tens, add the second figure of the root, considered as units; multiply their sum by the second figure of the root and subtract the product from the remainder.*

Reason. The purpose of the step is to find out what remains of the original power after subtracting from it the square of the root already found. This could be done by squaring the root and subtracting it from the power. The square of tens has, however, already been subtracted. There remains twice the tens by the units, plus the square of the units, to be subtracted; or, what is the same thing, the sum of twice the tens plus the units, multiplied by the units. This product, subtracted from the first remainder, gives the remainder sought.

If there are three or more periods in the power, the following addition to the rule as already given is necessary.

6. *In proceeding to find each new figure of the root, consider the root already found as tens and the figure sought as units. Then repeat steps 4 and 5.*

Let the square root of 568516. be required. The greatest square contained in 5685 may be found by the rule already explained for finding the square root of a number composed of two periods.

The work up to this point would stand as follows:

```
        56,85,16(75
        49
    145| 785
       | 725
         6016
```

Having now found the greatest square in the first two periods and subtracted it from these periods, we may regard the root already found as 75 tens and proceed to find the next term by repeating steps 4 and 5.

The entire work would then stand as follows:

```
                           tens   units.
                  56,85,16 | 75 +  4
                  49
     145           | 785
 tens    units       725
2×75+     4          | 6016
                     | 6016
```

By continuing this process, the square root of a quantity containing any number of periods can evidently be found.

EXAMPLES.

Find the square root of the following:

1. 183184.
2. 404291449.
3. 39.9424.
4. 8.1.
5. 39505.5376.
6. .00123.
7. 3 to 5 decimal places.
8. $\frac{4489}{90601}$.
9. $\frac{1}{2}$ to 4 decimal places.
10. $\frac{7}{128}$ to 4 decimal places.
11. $\frac{35}{8}$ to 4 decimal places.

210. Cube Root of Polynomials.

Let a and b represent any quantities,

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

The second member of the equation is the *power*; $a+b$ is the *root*. The power is equal to the cube of the first term of the root, plus three times the square of the first term of the root by the second, plus three times the first term of the root by the square of the second, plus the cube of the second term of the root.

The steps by which the cube is built up are thus made clear; reversing these steps, the root can be readily obtained from the power.

$$\begin{array}{lll} & \text{power} & \text{root} \\ & a^3+3a^2b+3ab^2+b^3 & \underline{|\,a+b} \\ & \underline{a^3} & \\ 3a^2+3ab+b^2 & |\,3a^2b+3ab^2+b^3 & \\ & \underline{|\,3a^2b+3ab^2+b^3} & \end{array}$$

Stating the process in the form of a rule and giving the reason for each step taken, we have the following:

1. *Arrange the terms according to the descending exponents of some one letter.*

Reason. For convenience, and because this is the order in which the terms naturally appear in the power when it is derived from the root by continuous multiplication.

2. *Extract the cube root of the first term of the power and it will give the first term of the root.*

Reason. The cube of the first term of the root equals the first term of the power; therefore the cube root of the first term of the power will give the first term of the root.

3. *Subtract the cube of the root already found from the power.*

Reason. By this step we obtain the remainder of the power. If the remainder is 0, it shows that the exact root has been found.

4. *Divide the first term of the remainder by three times the square of the first term of the root and it will give the second term of the root.*

Reason. In forming the power, its *second* term which is the *first* term of the remainder was obtained by multiplying three times the square of the first term of the root by the second; therefore, if we divide the first term of the remainder by three times the square of the first term of the root, it will give the second term of the root.

5. *Add to three times the square of the first term of the root three times the first term of the root by the second and the square of the second term of the root; multiply this sum by the second term of the root and subtract the product from the first remainder.*

Reason. The object of this step is to find what remains of the power after there has been subtracted from it, the cube of the entire root already found. This remainder might have been obtained by cubing the root and subtracting it from the power. Since, however, the cube of the first term of the root has been already subtracted, the desired result can be obtained by subtracting from the *first remainder* the remaining part of the cube of the root, viz., $3a^2b + 3ab^2 + b^3$. This is equivalent to subtracting from the first remainder $(3a^2 + 3ab + b^2)b$, which translated gives us step 5. (Student translate.)

If the remainder thus obtained is 0, the exact root has been found.

Note these various steps in the following example:

Find the cube root of $8a^3+36a^2b+54ab^2+27b^3$.

$$\begin{array}{r|l}
 & 8a^3+36a^2b+54ab^2+27b^3 \;|\,\underline{2a+3b} \\
 & 8a^3 \\ \hline
12a^2+18ab+9b^2 & 36a^2b+54ab^2+27b^3 \\
 & 36a^2b+54ab^2+27b^3
\end{array}$$

If there are three monomial terms in the root, the power may still be regarded as the cube of a binomial; for representing these terms by a, b and c, the root may be put in the form $(a+b)+c$.

$[(a+b)+c]^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$.

By the steps already given, the first binomial term of the root, $(a+b)$ can be obtained.

The work will then stand as follows:

$$\begin{array}{l}
(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3 \;|\,\underline{(a+b)} \\
(a+b)^3 \\ \hline
\qquad 3(a+b)^2c+3(a+b)c^2+c^3.
\end{array}$$

The remainder of the work will be a repetition of steps 4 and 5, simply regarding and speaking of $(a+b)$ as the first term.

If there are more than three monomial terms in the root, the work connected with the finding of each of the remaining terms of the root will be a mere repetition of the steps 4 and 5, regarding in each case all previous terms of the root as *one term*.

EXAMPLES.

Find the cube root of the following:

1. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
2. $3b^5+b^6-5b^3-1+3b$.

✓ 3. $27x^6 - 54x^5 + 9x^4 + 28x^3 - 3x^2 - 6x - 1.$

4. $132x^4 - 182x^3 + 27x^5 + 1 + 6x^8 + 81x^2 +$
$x^9 - 15x - 3x^7 - 49x^6.$

✓ 5. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 +$
$10x^6 - 6x^7 + 3x^8 - x^9.$

6. $m^6 + 9m^5y - 135m^3y^3 + 729my^5 - 729y^6.$

7. $8x^6y^3 + 60x^4cy^2z - 12x^5y^2 + 125c^3z^3 + 150x^2yc^2z^2$
$- x^3 - 75c^2xz^2 + 6x^4y + 15cx^2z - 60x^3ycz.$

8. $8a^{\frac{3}{2}} - 12a^{\frac{1}{2}} + 42a^{-\frac{1}{2}} - 37a^{-\frac{3}{2}} + 63a^{-\frac{5}{2}} -$
$27a^{-\frac{7}{2}} + 27a^{-\frac{9}{2}}.$

9. $(x+a)^{6P}y^3 - 6dy^2bm(x+a)^{4P} +$
$12d^2b^{2m}(x+a)^{2P}y - 8d^3b^{3m}.$

10. $1 - x$, finding three terms.

211. Cube Root of Numbers.

The cube of any number is built up in the same way as the cube of a polynomial, thus,

(tens + units)^3 = tens^3 + 3 × tens^2 × units + 3 × tens × units^2 + nnits^3.

$$(54)^3 = (50+4)^3 = (50)^3 + 3\,(50)^2 \times 4 + 3 \times 50 \times 4^2 + 4^3 = 15764.$$

As in the square of a number, so in the cube, the terms of which it is composed are blended together. They must be separated in order to find the root. This process is called *separating the number into periods.*

$2^3 = 8$ $(20)^3 = 8000$ $.2^3 = .008$
$3^3 = 27$ $(30)^3 = 27000$ $.3^3 = .027$
$9^3 = 729$ $(90)^3 = 729000$ $.9^3 = .729$

It is evident from the above that if the cube be separated into periods of three figures each, beginning at the decimal point, the cube of the unit figure in the root will be found in the first period to the left of the decimal point, the cube of tens in the second period, and so on; while the cube of tenths will be found in the first period to the right of the decimal point, the cube of hundredths in the second period, and so on. It follows that for each period in the power there will be a figure in the root, and that each period and the corresponding root figure will be similarly situated with reference to the decimal point.

Divide into periods the following:

1. 526435.
2. 7425329.
3. 4329.546.
4. 29.5.
5. .35274.
6. $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{2}$, after reducing each to decimal form.

After separating the power into periods, the remaining steps in extracting the cube root of numbers are the same as in extracting the cube root of algebraic polynomials.

The process, when there are two figures in the root, will be as follows:

$$\begin{array}{r|l}
 & \text{tens}^3+3\times\text{tens}^2\times\text{units}+3\times\text{tens}\times\text{units}^2+\text{units}^3 \;\big|\; \underline{\text{tens}+\text{units}} \\
 & \underline{\text{tens}^3} \\
3\times\text{tens}^2+3\times\text{tens}\times\text{units}+\text{units}^2 & 3\times\text{tens}^2\times\text{units}+3\times\text{tens}\times\text{units}^2+\text{units}^3 \\
 & \underline{3\times\text{tens}^2\times\text{units}+3\times\text{tens}\times\text{units}^2+\text{units}^3}
\end{array}$$

Let it be required to extract the cube root of 13,824.

$$\begin{array}{lrl}
 & & 13{,}824 \;\big|\; \underline{24} \\
 & & \quad 8 \\
3\times\text{tens}^2=3\,(20)^2=3\,(400)= & 1{,}200 & \big|\; \overline{5824} \\
3\times\text{tens}\times\text{units}=3\,(20)\,(4)= & 240 & \big| \\
\text{units}^2=4^2 \qquad\qquad = & \underline{16} & \big| \\
 & 1{,}456 & \big|\; \underline{5824}
\end{array}$$

The steps taken, with the reason for each, can be stated as follows:

1. *Separate the number into periods of three figures each, beginning with the decimal point.*

Reason. We thus obtain the number of figures in the root and their position with reference to the decimal point.

2. *Extract the cube root of the greatest cube in the left-hand period of the power and it will give the first figure, tens, in the root.*

Reason. The cube of the first figure of the root, tens, appears only in the left-hand period of the power; therefore, if we extract the cube root of the greatest cube in the left-hand period of the power, it will give the first figure of the root.

3. *Subtract the cube of the first figure of the root from the left-hand period of the power and bring down the next period.*

Reason. This will give what remains of the power after we have subtracted from it the cube of the root already found.

4. *Divide this remainder by three times the square of the first figure of the root, regarded as tens, and it will give the second figure of the root.*

Reason. The remainder contains three times the square of the tens by the units, plus three times the tens by the square of the units, plus the cube of the units. These terms cannot be separated, but the first is much the greatest, hence it follows that if we divide this remainder by three times the square of the tens we will probably obtain the units. As this division does not, however, always give the units correctly, three times the square of the tens is called the *trial divisor.* If the first trial with this divisor leads to a wrong unit figure for the root, the subsequent work will reveal it, and the error can be corrected.

5. *To three times the square of the tens, add three times the tens by the units and the square of the units, multiply this sum by the second or unit figure of the root and subtract the product from the first remainder.*

Reason. By this step we seek to find what remains of the power after subtracting from it the cube of the entire root found. The cube of the tens was subtracted in a previous step. We now subtract the remainder of the cube which factored consists of $(3 \times \text{tens}^2 + 3 \times \text{tens} \times \text{units} + \text{units}^2) \times \text{units}$. Hence the step as stated.

If there are more than two figures in the root, we may, in finding the third or any succeeding figure, regard the previous figures as *tens* and repeat steps 4 and 5.

Thus, in extracting the cube root of 44,361,864, we may at first deal simply with the first two periods, 44,361, and find the first two figures of the root 35. Subtracting the cube of 35 from these periods and bringing down the next period, we will have as our remainder 1486864. In finding the next figure of the root we may regard 35 as *tens*, and the figure sought as units. The following shows in full the work of finding the third term:

$$
\begin{array}{lllr|l}
 & & & & \text{tens} \quad \text{units} \\
 & & & 44361864 & \underline{35 \;+\; 4} \\
 & & & \underline{42875} & \\
3\times\text{tens}^2 & = 3\times(350)^2 & = 367500 & 1486864 & \\
3\times\text{tens}\times\text{units} & = 3\times350\times4 & = 4200 & & \\
\text{units}^2 & = \quad 4^2 & = \underline{16} & & \\
 & & 371716 & \underline{1486864}. &
\end{array}
$$

EXAMPLES.

Find the cube root of the following:

1. 77854483.
2. 1070599167.
3. $\frac{17576}{79507}$
4. 39.651821.
5. .08 to four decimal places.
6. 2. to four decimal places.
7. $\frac{2}{3}$ to four decimal places.
8. $\frac{13}{4}$ to three decimal places.

212. Higher Roots of Polynomials.

The rules for extracting the higher roots of polynomials are derived in the same way as are those for finding the square and cube root of such quantities.

A careful examination into the manner in which the *power* is built up from the *root* will, in each case, reveal the steps by which the root can be found, when the power is given.

The student should bear in mind that there are two ways of finding the *remainders*. He has become acquainted with one of these, in the work in square and cube root; the other, which if used at all must of course be used as often as a new term in the root is obtained, consists in raising the entire root found up to that point, to the required power and subtracting the result from the original power.

EXERCISES.

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$

Give and explain fully each step taken in extracting the following roots:

1. The fourth root of algebraic polynomials.
2. The fifth root of algebraic polynomials.
3. The sixth root of algebraic polynomials.
4. The fourth root of numbers.
5. The fifth root of numbers.
6. The sixth root of numbers.

SUGGESTIVE QUERIES.

1. In finding the different terms of any higher root of algebraic polynomials, what part of the divisor in each case is used?

2. What relation exists between the parts used of the different divisors?

3. What is meant by 1st remainder, 2d remainder, etc.?

4. Which of the two methods of obtaining the different remainders involves the least work?

5. By what other method can the fourth and sixth roots be found?

EXAMPLES.

1. Find the fourth root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

2. Find the fourth root of 13107.9601.

3. Find the fifth root of $x^{10} - 10x^9 + 30x^8 - 120x^6 + 48x^5 + 240x^4 - 240x^2 - 160x - 32$.

4. Find the fifth root of 36936242722357.

5. Find the sixth root of $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

6. Find the sixth root of .009474296896.

CHAPTER XVI.

RADICALS.

213. It was stated in evolution:

1. That any root of a quantity may be indicated by the radical sign and its index.

Thus, $\sqrt{a}$, $\sqrt[3]{ab}$, $\sqrt[5]{5ac}$, $\sqrt[n]{2abc}$ are indicated roots.

2. That when the root can be exactly extracted, the quantity is called a *rational quantity*.

3. That when the root cannot be exactly extracted, the quantity is called an *irrational* or *surd quantity*.

214. The **degree** of a radical is indicated by the index of the root.

Thus, $\sqrt[n]{a}$ is a radical of the nth degree.

215. An **entire radical** is one that has no rational factor outside the radical sign.

216. A **mixed radical** is the product of a rational factor and a surd factor.

Thus, $3a\sqrt{bc}$ and $4\sqrt[3]{2a}$ are mixed radicals.

217. **Like radicals** are those which have the same radical sign and the same quantity under the radical.

Thus, $3a\sqrt[3]{2b}$ and $2c\sqrt[3]{2b}$ are like radicals.

In like mixed radicals, the radical is always a common factor.

218. It was shown in evolution that any root may be indicated by a fractional exponent. It follows:

1. That any radical quantity may be changed to an equivalent expression in which a fractional exponent is used.

2. That any expression with a fractional exponent may be put in the radical form.

EXAMPLES.

Change the following to equivalent expressions with fractional exponents:

1. $\sqrt{a}$.
2. $\sqrt[4]{m^3}$.
3. $\sqrt{x^m}$.
4. $\sqrt[n]{a^m}$.
5. $\sqrt[5]{c^{-m}}$.
6. $\sqrt{\frac{1}{a^4}}$.
7. $\sqrt[n]{\frac{1}{a}}$.
8. $\sqrt{\frac{1}{x^{-2}}}$.
9. $\sqrt{abc}$.
10. $\sqrt[3]{cde^n}$
11. $\sqrt[p]{c^m a^n b^n}$.
12. $\sqrt[n]{a+b}$.
13. $\sqrt[m]{c+d}$.
14. $\sqrt[5]{\frac{1}{c+d}}$.
15. $\sqrt{\frac{1}{(x-a)^{-3}}}$.

Change the following to equivalent expressions in radical form:

1. $a^{\frac{1}{3}}$.
2. $x^{\frac{3}{5}}$.
3. $y^{\frac{1}{m}}$.
4. $a^{\frac{2+m}{3}}$.
5. $a^{\frac{x^2}{7}}$.
6. $a^{-\frac{1}{2}}$.
7. $a^{-\frac{1}{n}}$.
8. $a^{-\frac{m}{n}}$.
9. $x^{\frac{1}{n}}y^{\frac{2}{n}}z^{\frac{3}{n}}$.
10. $m^{\frac{5}{2}}n^{\frac{3}{2}}p^{\frac{1}{2}}$.
11. $a^{\frac{1}{4}}b^{\frac{3}{4}}c^{\frac{1}{2}}$.
12. $(a+b)^{\frac{1}{3}}$.
13. $(x-y)^{\frac{m}{n}}$.
14. $(c+d)^{\frac{2}{5}}(x-y)^{\frac{3}{5}}$.
15. $(x-a)^{-\frac{2}{3}}$.

219. Since an integer may be reduced to an equivalent fraction having any desired denominator, it is clear that an algebraic quantity with an integral exponent, may be put in the form of an equivalent radical of any desired degree,

Thus,

$$a = a^{\frac{2}{2}} = \sqrt{a^2} \qquad ab = (ab)^{\frac{2}{2}} \qquad = \sqrt{(ab)^2} = \sqrt{a^2b^2}$$

$$a = a^{\frac{3}{3}} = \sqrt[3]{a^3} \qquad ab = (ab)^{\frac{n}{n}} \qquad = \sqrt[n]{(ab)^n} = \sqrt[n]{a^nb^n}$$

$$a = a^{\frac{n}{n}} = \sqrt[n]{a^n} \qquad (a+b) = (a+b)^{\frac{n}{n}} = \sqrt[n]{(a+b)^n}$$

EXAMPLES.

1. Reduce ax^2 to a radical of the second degree.
2. Reduce $2a^3bx^2$ to a radical of the third degree.
3. Reduce -2 to a radical of the third degree.
4. Reduce $3a - b$ to a radical of the second degree.
5. Reduce $\frac{x^2}{3}$ to a radical of the third degree.
6. Reduce $\frac{abc}{x-y}$ to a radical of the fourth degree.

220. It is evident from the foregoing that *all algebraic quantities* can be reduced to the radical form, and that in this form they are subject to the same laws as when the quantities are expressed by the aid of integral and fractional exponents.

It follows that:

The rules of operation for radical quantities are the same as those for quantities not expressed in the form of radicals.

REDUCTION OF RADICALS.

221. **To reduce a radical** is to change its form without altering its value.

222. Preparatory to considering particular methods of reduction, the student is required to state in review:

1. The first rule for raising a factor of a quantity to any power.

2. The second rule for raising a factor of a quantity to any power.

3. The first rule for extracting any root of a factor of a quantity.

4. The second rule for extracting any root of a factor of a quantity.

223. These rules when applied to factors consisting of bases affected with fractional exponents, give rise to the following:

(*a*) **Rules for raising a factor to any power.**

1. *Multiply the numerator of the fractional exponent of the base by the exponent of the power.*

or 2. *Divide the denominator of the fractional exponent of the base by the exponent of the power.*

or 3. *Raise the base to the required power and leave its exponent unchanged.*

EXAMPLES.

1. Raise to the second power, by each of the three rules:

$2^{\frac{3}{4}}$. $(a^2)^{\frac{1}{10}}$. $(3a)^{\frac{5}{8}}$. $(a+b)^{\frac{5}{6}}$.

2. Raise to the third power, by each of the three rules:

$$3^{\frac{2}{3}}. \qquad (a^3)^{\frac{1}{6}}. \qquad (4x)^{\frac{2}{9}}. \qquad (x+y)^{\frac{1}{2}}.$$

In the following, choose the rule most convenient:

3. Required the second power of $2^{\frac{1}{3}}$.

4. Required the third power of $(a^3)^{\frac{2}{5}}$.

5. Required the fourth power of $\frac{2}{3}a^{\frac{4}{3}}$.

6. Required the second power of $(a+b)^{\frac{1}{4}}$.

(*b*) Rules for Extracting any Root of a Factor.

1. *Divide the numerator of the fractional exponent of the base by the index of the required root.*

or 2. *Multiply the denominator of the fractional exponent of the base by the index of the required root.*

or 3. *Extract the required root of the base and leave its exponent unchanged.*

EXAMPLES.

1. Extract the square root of the following, by each of the three rules:

$$4^{\frac{2}{5}}. \qquad (a^4)^{\frac{4}{3}}. \qquad (9x^2)^{\frac{6}{7}}. \qquad (a^2+2ab+b^2)^{\frac{2}{3}}.$$

2. Extract the cube root of the following, by each of the three rules:

$$(27)^{\frac{3}{4}}. \quad (a^6)^{\frac{9}{10}}. \quad (8x^3)^{\frac{6}{7}}. \quad (x^3+3x^2y+3xy^2+y^3)^{\frac{3}{5}}.$$

In the following, choose the rule most convenient:

3. Required the cube root of $(3a^2)^{\frac{1}{3}}$.

4. Required the fourth root of $(81a^4)^{\frac{1}{5}}$.

5. Required the square root of $(a^2-b^2)^{\frac{1}{2}}$.

224. Remembering that the index of the radical corresponds to the denominator of the fractional exponent, these rules of involution and evolution, when applied to radicals, can be stated as follows:

(*a*) Rules for Raising to any Power.

1. *Multiply the exponent of the basal quantity under the radical sign, by the exponent of the power.*

or 2. *Divide the index of the radical by the exponent of the power.*

or 3. *Raise the basal quantity under the radical to the required power, and leave its exponent and the index of the radical unchanged.*

EXAMPLES.

Find the required power of the following, by each of the three rules:

1. The 3d power of $\sqrt[6]{(2a)^5}$.

2. The 5th power of $\sqrt[10]{(3m^2n)^7}$.

3. The 2d power of $\sqrt[8]{(x+y)^3}$.

4. The nth power of $\sqrt[mn]{(4a^2b)^y}$.

Find the required power of the following, by the rule which is most convenient:

1. The 4th power of $\sqrt{a+x}$.

2. The 2d power of $\sqrt[5]{x^2+2xy+y^2}$.

3. The 3d power of $\sqrt[5]{(m-n)^{12}}$.

4. The nth power of $\sqrt{2^n a^{ny} f^{n^2}}$.

(*b*) **Rules for Extracting any Root.**

1. *Divide the exponent of the basal quantity under the radical sign, by the index of the root.*

or 2. *Multiply the index of the radical by the index of the root.*

or 3. *Extract the required root of the basal quantity under the radical sign, and leave its exponent and the index of the radical unchanged.*

EXAMPLES.

Extract the required root of the following, by each of the three rules:

1. The square root of $\sqrt[3]{(x^2+2xy+y^2)^4}$.

2. The cube root of $\sqrt[5]{(8a^3b^6)^{12}}$.

3. The 5th root of $\sqrt[n]{(32a^{10}b^5)^{5y}}$.

4. The nth root of $\sqrt{(3^n a^{nx} y^{n^3})^{4n}}$.

Extract the required root of the following by the rule which is most convenient:

1. The cube root of $\sqrt[5]{x+y}$.

2. The 5th root of $\sqrt{(2m^2n)^{15}}$.

3. The square root of $\sqrt[9]{4x^2(m^2-2mn+n^2)}$.

4. The nth root of $\sqrt{[(2^n(a+x)^{nx}]^y}$.

225. It is evident that if a quantity with a fractional exponent be raised to a power by any one of the three rules of involution, and the root of equal degree, extracted by any one of the three rules of evolution, the final result will have the same value as the original quantity.

The order in which these steps are taken is a matter of convenience to be decided by the student.

EXAMPLES.

Change the form of each of the following, without altering its value:

1. $(6)^{\frac{2}{4}}$.
2. $(4a^2)^{\frac{3}{5}}$.
3. $(16a^4m^{8x})^{\frac{1}{7}}$.
4. $[(x^2+2xy+y^2)z^4m^6]^{\frac{3}{4}}$.
5. $[a^nx^{2n}(m-n)^{4n}]^{\frac{1}{2}}$.
6. $[4ax^2y(m+z)]^{\frac{2}{3}}$.

226. Quantities expressed in the radical form can, in a similar way, under the rules given, be raised to any power and the corresponding root then extracted. Radical quantities can thus be changed in form without altering their value. This method of reducing radicals gives rise to the following cases:

Case I. *In which the degree of the radical is not changed.*

EXAMPLES.

Involving the multiplication of the exponent of the basal quantity:

1. $\sqrt[3]{(4a^2b^2c^2)^2}$. Ans. $\sqrt[3]{(2abc)^4}$.
2. $\sqrt[5]{(8a^3b^9c^{12})^4}$.
3. $\sqrt{(32a^{10}b^5c^{15})^n}$.
4. $\sqrt[m]{\left(\frac{8}{27}a^9x^3y^{12}\right)^4}$.
5. $\sqrt[r]{\left[\frac{9}{16}c^8b^2(x+y)^4\right]^{-6}}$.

6. $\sqrt[5]{\left[\frac{4}{25}m^6n^8(a^2+2ab+b^2)\right]^s}$.

7. $\sqrt[3]{a^2-2ab+b^2}$.

8. $\sqrt[n]{a^6(x^3+3x^2y+3xy^2+y^3)}$.

9. $\sqrt[7]{\left(\frac{16}{81}x^{-4}y^{8n}z^{12m}\right)^p}$. 10. $\sqrt[5]{(x^{3n}y^{bn}z^{(x-y)n})^2}$.

11. $\sqrt[m]{[x^n y^{n^2} c^{bn^3} (a-b)^{-2n^2}]^3}$.

Involving the division of the exponent of the basal quantity:

1. $\sqrt[5]{(2ab^2c)^6}$.

Ans. $\sqrt[5]{(4a^2b^4c^2)^3}$, or $\sqrt[5]{(8a^3b^6c^3)^2}$, or $\sqrt[5]{64a^6b^{12}c^6}$.

2. $\sqrt[3]{(a^2b^4c^8)^{2n}}$. 3. $\sqrt[3]{(x^6y^9z^{3m})^{2n}}$.

4. $\sqrt[7]{(xy^2(a+b))^6}$. 5. $\sqrt{[x^my^{3m}(c+d)^{m^2}]^m}$.

6. $\sqrt{\left[\frac{2}{3}ab^{-1}x^{-2n}(x+y)^n\right]^{2n}}$.

7. $\sqrt{[3^{-1}d^{2n}(a+b)^{x+y}]^{mn}}$.

Case II. *In which the degree of the radical is multiplied.*

EXAMPLES.

Reduce each of the following to equivalent radicals of degrees which shall be equal to the degree of the original radical multiplied by 2, 3, 4, and n respectively.

1. $\sqrt{(2ab)^x}$. Ans. for 4th degree $\sqrt[4]{(2ab)^{2x}}$ or $\sqrt[4]{(4a^2b^2)^x}$.

2. $\sqrt[3]{3ab^2}$. 3. $\sqrt[5]{x+y}$.

4. $\sqrt[3]{2a^2bx(m+n)}$. 5. $\sqrt{3a^nx^{-1}}$.

6. $\sqrt[3]{4bx^2(x-a)^2}$.

7. $\sqrt[5]{-3abm^{-2}(a+x)^{-3}\cdot(a-x)^m}$.

Reduction of Radicals to a Common Index. The reduction of radicals of different degrees to equivalent radicals of the same degree comes under this case.

EXAMPLES.

Reduce to a common index the following:

1. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$. Ans. $\sqrt[12]{a^6}$, $\sqrt[12]{a^4}$, $\sqrt[12]{a^3}$.

2. $\sqrt{2ab}$, $\sqrt[3]{3ab^2}$, $\sqrt{5ab^3}$.

3. $\sqrt[4]{a^n}$, $\sqrt[6]{-5b}$, $\sqrt[8]{-3c^m}$.

4. $\sqrt{2ax}$, $\sqrt[m]{ay}$, $\sqrt[n]{bx}$.

5. a, $\sqrt{ac}$, $\sqrt[3]{a^2x}$, $\sqrt[4]{2ac^2}$.

6. $\sqrt{a-x}$, $\sqrt[3]{a^2-x^2}$.

7. $\sqrt{2a(a+b)}$, $\sqrt[4]{a-b}$.

8. $\sqrt{a+x}$, $\sqrt[3]{(a-x)^2}$.

9. $\sqrt{x-y}$, $\sqrt[3]{x+y}$, $\sqrt[6]{x^2-y^2}$.

10. $\sqrt{2a(m^2-n^2)}$, $\sqrt[4]{5c(m^2+n^2)}$, $\sqrt[7]{2b(m-n)}$.

11. $\sqrt{x-y}$, $\dfrac{1}{\sqrt[4]{x+y}}$.

Where the radical is preceded by a coefficient, this coefficient should not be changed.

12. $2c\sqrt{x}$, $5a\sqrt[3]{2z}$.

13. $a\sqrt{a-x},\ b\sqrt[3]{c+x}$.

14. $2b\sqrt{a^2x^3z^{-4}},\ 3c\sqrt[5]{bx^2z}$.

15. $4\sqrt[3]{5cx^2},\ 2\sqrt[6]{2xm},\ 10a\sqrt{3bx}$.

Case III. *In which the degree of the radical is divided.*

In this case the resulting radical with the lower index is considered to be in *simpler form* than the original radical.

EXAMPLES.

1. $\sqrt[6]{(16a^2b^4)^{2x}}$ Ans. $\sqrt[3]{(4ab^2)^{2x}}$ or $\sqrt[3]{(16a^2b^4)^x}$

2. $\sqrt[10]{4a^2}$

3. $\sqrt[4]{81m^{-2}c^4}$

4. $\sqrt[8]{(25x^2)^2}$

5. $\sqrt[4]{\frac{4}{9}}$

6. $\sqrt[6]{\frac{16}{25}}$

7. $\sqrt[24]{\left(\frac{25}{64}\right)}$

8. $\sqrt[4]{\frac{25a^2}{36x^2}}$

9. $7\sqrt[9]{-8a^3}$

10. $3a\sqrt[4]{x^2-2xy+y^2}$

11. $(a-x)\sqrt[mn]{a^nb^{2n}}$

12. $\sqrt[4]{\frac{(x-5)^2}{(x+3)^2}}$

13. $\sqrt[6]{\frac{8a^3b^6(x+y)^3}{27x^3y^3(x-y)^9}}$

14. $\sqrt[4]{\frac{25}{64}a^2b^{-6}c^8d^{4n}}$

15. $\sqrt{4a^2+12ab+9b^2}$

16. $\sqrt{a^{2n}b^{-4}c^6(x-y)^{-2}}$

17. $\sqrt[6]{(8a^3b^3)^m}$

18. $\sqrt[9]{-27a^6(x+y)^{-6}}$

19. $\sqrt{(4a^2+4a+1)^5}$

Reduction of Radicals to their Simplest Form.

227. A radical is in its simplest form when it is of the lowest possible degree, and has under the radical sign, no factor which is a perfect power of the same degree as the reduced radical.

228. Case I. *In which the degree of the radical is decreased.*

This is the same as *case III* under reductions effected by the rules of involution and evolution.

229. Case II. *In which the radical is separated into two factors and the root of one of the factors extracted.*

This is simply doing in part what cannot be done as a whole. If the exact root of the entire quantity could be obtained, it would be well; since the root of one of the two factors can be extracted, this much is done and the root of the second factor indicated. One of the factors should be the highest perfect power of the degree corresponding to the index of the root. In separating into factors, the radical sign of proper degree should be placed over each factor.

EXAMPLES.

1. $\sqrt{125a^3} = \sqrt{25a^2}\,\sqrt{5a} = 5a\sqrt{5a}$.

2. $\sqrt{294ab^2}$.

3. $7\sqrt{80ab^3}$.

4. $\sqrt{a^3 - a^2b}$.

5. $\sqrt{ax^2 - 6ax + 9a}$.

6. $\sqrt{8a^{2n}x^{4n}}$.

7. $\sqrt{(x^2 - y^2)(x+y)}$.

8. $\sqrt[3]{56a^5b^6}$.

9. $9\sqrt[3]{-54a^4b^3c^2}$.

10. $\sqrt[3]{7290x^{8m}y^{6n+2}}$.　　11. $\sqrt[3]{1715a^{9mn}b^{6n}}$.

12. $2c\sqrt[4]{768}$.　　13. $6\sqrt[5]{a^{18}c^8}$.

14. $\sqrt[5]{x^{10}y^{-5}z^{5m+1}}$.

15. $(a-b)\sqrt{2a^2b+4ab^2+2b^3}$.

16. $\sqrt[n]{a^{m+n}b^n}$.

17. $a^{-m}c\sqrt[n]{a^{mn}c^{2n}-a^{2mn}c^n}$.

A fraction, under the radical sign, should be reduced to an equivalent fraction whose denominator is a perfect power of the same degree as the radical, before the quantity under the radical is separated into factors. Care should be taken to resolve the numerator of the fraction into its proper factors.

Illustrative Example.

$$\sqrt{\frac{4}{5}}=\sqrt{\frac{20}{25}}=\sqrt{\frac{4}{25}}\sqrt{5}=\frac{2}{5}\sqrt{5}.$$

EXAMPLES.

1. $3\sqrt{\frac{2}{5}}$.　　2. $\frac{3}{4}\sqrt{\frac{5}{6}}$.

3. $\sqrt{8\frac{1}{3}}$.　　4. $a\sqrt{\frac{b}{a}}$.

5. $(a-b)\sqrt{\frac{a+b}{a-b}}$.　　6. $\frac{a^3b^2}{c^2}\sqrt{\frac{c^5}{a^5b^5}}$.

7. $\frac{x}{ay}\sqrt{\frac{a^3cy^2}{x^3}}$.　　8. $\sqrt{\frac{ax^2}{4(a-x)}}$.

9. $\sqrt[5]{\frac{512m^2}{125n^2}}$.

10. $\sqrt[3]{\frac{(x-a)z^5}{(x+a)^2x}}$.

11. $\frac{a}{a^2-b^2}\sqrt{\frac{a^3-2a^2b+ab^2}{b^3}}$.

12. $3\sqrt[4]{\frac{x^5z^7}{8}}$.

13. $5n\sqrt[n]{\frac{a^{n+1}b^{n-1}}{n^{n-1}}}$.

ADDITION OF SURDS.

230. The addition of surd quantities, whether expressed in radical form or by fractional exponents, presents no new principles.

The surds to be added should first be reduced to their simplest form; the remainder of the work is the same as in the addition of rational quantities.

Hence the following

RULE.

Reduce each surd to its simplest form, prefix to the highest common factor of the resulting surds, the algebraic sum of its coefficients. If there is no common factor, connect the surds by their proper signs.

EXAMPLES.

Find the sum of the following:

1. $bx^{\frac{1}{2}}$ and $ax^{\frac{1}{2}}$.

2. $abx^{\frac{1}{3}}$ and $(c+d)\,x^{\frac{1}{3}}$.

3. $(2+m)\,(a-b)^{\frac{1}{5}}$ and $(a-b)^{\frac{1}{5}}$.

4. $cy^{\frac{3}{2}}$ and $dy^{\frac{1}{2}}$.

5. $(m+n)(m-n)^{\frac{3}{2}}$, $(m-n)^{\frac{5}{2}}$ and $2mn(m-n)^{\frac{1}{2}}$.

6. $\frac{x}{4}(x^2+a^2)^{\frac{3}{2}}$ and $\frac{3a^2x}{8}(x^2+a^2)^{\frac{1}{2}}$

7. $\frac{2x^6}{15}(1-x^3)^{\frac{1}{2}}$, $\frac{2x^3}{45}(1-x^3)^{\frac{3}{2}}$ and $\frac{1}{45}(1-x^3)^{\frac{5}{2}}$.

8. $\sqrt{27}$, $\sqrt{48}$ and $\sqrt{75}$.

9. $4\sqrt{72}$, $3\sqrt{128}$, and $\sqrt{162}$.

10. $\sqrt[m]{a^mb}$, and $\sqrt[m]{c^mb}$.

11. $\sqrt[3]{24}$ and $\sqrt[3]{81}$.

12. $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.

0 13. $6\sqrt[6]{4a^2}$, $2\sqrt[3]{2a}$ and $\sqrt[9]{8a^3}$.

14. $2\sqrt{\frac{3}{7}}$, $\frac{1}{3}\sqrt{84}$, $\sqrt{21}$, and $\sqrt{\frac{7}{3}}$.

15. $2c\sqrt{m^2y-m^2z}$, $4m\sqrt{c^2y-c^2z}$ and
$3\sqrt{c^2m^2y-c^2m^2z}$.

16. $\sqrt{45a^2m-120abm+80b^2m}$ and
$\sqrt{80b^2m-40bm+5m}$.

17. $\sqrt{\frac{x^2(x-a)}{x+a}}$, $\sqrt{\frac{a^2(x+a)}{x-a}}$ and $(2a^2-4x^2,\sqrt{\frac{1}{x^2-a^2}}$.

18. $\sqrt[x]{m^{x+1}c}$ and $\sqrt[x]{mcy^{2x}z^{x^2}}$.

19. $\sqrt[3]{54x^{a-3}y^6}$, $\sqrt[3]{16x^{a+6}y^3}$ and $\sqrt[3]{2x^{4a-9}}$.

20. $\sqrt[y]{4^yb^{py+3}c^{ny+5}}$, $\sqrt[y]{3^yb^{3y-ny+3}c^{y+5}}$ and $\sqrt[y]{2^yb^3c^5x^{2y}}$.

21. $\sqrt{\frac{5a^{m+2}b^3}{x^2}}$ and $\frac{1}{2x}\sqrt{20a^mb^5 - 80a^{m+1}b^4 + 80a^{m+2}b}$.

Subtraction of Surds.

231. Subtraction of surds does not differ in principle from subtraction of rational quantities.

We have then the following

RULE.

Conceive the signs of all the terms of the subtrahend to be changed and proceed as in addition of surds.

EXAMPLES.

1. From $ax^{\frac{1}{3}}$ subtract $bx^{\frac{1}{3}}$.

2. From $b(m-n)^{\frac{1}{5}}$ substract $cx(m-n)^{\frac{1}{5}}$.

3. From $(a^2c^3 - 3c^3x)^{\frac{1}{3}}$ subtract $2(a^2d^3 - 3d^3x)^{\frac{1}{3}}$.

4. From $(a^3 - ab^2 + a^2b - b^3)^{\frac{1}{2}}$ subtract $(a^3 - 3a^2b + 3ab^2 - b^3)^{\frac{1}{2}}$.

5. From $\sqrt{448}$ subtract $\sqrt{112}$.

6. From $2\sqrt{50}$ subtract $\sqrt{18}$.

7. From $2\sqrt[3]{320}$ subtract $3\sqrt[3]{40}$.

8. From $\sqrt{63a^2x - 84abx + 28b^2x}$ subtract $\sqrt{7a^2x + 42abx + 63b^2x}$.

9. From $\frac{3}{8}\sqrt[3]{\frac{704}{27}}$ subtract $\frac{4}{3}\sqrt[3]{\frac{1375}{64}}$

10. From $y\sqrt[3]{\frac{27x^4y}{16b}}$ subtract $x\sqrt[3]{\frac{xy^4}{54b}}$

11. From $\frac{3}{7}\sqrt{\frac{245}{169}}$ subtract $\frac{5}{2}\sqrt{\frac{441}{5}}$

12. From $a\sqrt[m]{\frac{b^mx+b^m}{(x-1)^{m-1}}}$ subtract $b\sqrt[m]{\frac{a^mx-a^m}{(x+1)^{m-1}}}$

MULTIPLICATION OF SURDS.

Monday.

232. The general rules for the multiplication of monomials, govern in the multiplication of monomial surds.

Let it be required, in review, to state:

1. The *first rule* for the multiplication of the factors of monomial quantities.

2. The *second rule* for the multiplication of the factors of monomial quantities.

Monomial surds are either entire or mixed. In multiplication, the latter case includes the former. In its solution, the rational and the surd factors should be multiplied together separately. The product of the surd factors can always be found by the *second rule* for the multiplication of the factors of monomials; for under principles already established, these surds can be reduced to equivalent expressions with common fractional exponents, or, in the radical form, to a common index.

Illustrative Example.

Find the product of $(3ab)^{\frac{1}{2}}\times(2cd)^{\frac{1}{3}}$.

$$(3ab)^{\frac{1}{2}}=(3ab)^{\frac{3}{6}}=[(3ab)^3]^{\frac{1}{6}}=(27a^3b^3)^{\frac{1}{6}}.$$

$$(2cd)^{\frac{1}{3}}=(2cd)^{\frac{2}{6}}=[(2cd)^2]^{\frac{1}{6}}=(4c^2d^2)^{\frac{1}{6}}.$$

$$\text{Product,}\qquad=(108a^3b^3c^2d^2)^{\frac{1}{6}}.$$

The foregoing example, written in the radical form would read as follows:

Find the product of $\sqrt{3ab} \times \sqrt[3]{2cd}$.

$$\begin{aligned} \sqrt{3ab} &= \sqrt[6]{27a^3b^3}. \\ \sqrt[3]{2cd} &= \underline{\sqrt[6]{4c^2d^2}} \\ \text{Product} &= \sqrt[6]{108a^3b^3c^2d^2} \end{aligned}$$

EXAMPLES.

Find the value of the following expressions:

1. $5a^{\frac{1}{2}} \times 3a^{\frac{1}{3}}$.

2. $3x^{\frac{1}{2}}y^{\frac{1}{3}} \times 4x^{\frac{1}{2}}y^{\frac{2}{3}}$.

3. $(4a^2)^{\frac{1}{3}} \times (2a)^{\frac{2}{3}}$.

4. $(27x^3)^{\frac{1}{2}} \times (2y)^{\frac{3}{2}}$.

5. $(xy)^{\frac{1}{2}} \times (xz)^{\frac{2}{3}} \times (yz)^{\frac{3}{4}}$.

6. $(x-y)^{\frac{2}{3}} \times (x+y)^{\frac{3}{4}}$.

7. $3\sqrt{8} \times 2\sqrt{6}$.

8. $\sqrt{a-x} \times \sqrt{a+x}$.

9. $\sqrt{2} \times \sqrt[3]{3}$.

10. $2ax\sqrt[3]{4a^2} \times 3bx\sqrt{2a}$.

11. $5a\sqrt{3a} \times b\sqrt[3]{2a} \times \sqrt[3]{\frac{1}{6}a}$.

12. $4a\sqrt{ac} \times 3c\sqrt[3]{3cb} \times 2a\sqrt[4]{abc}$.

13. $\frac{ax+b}{y}\sqrt{\frac{ax}{b}}\times\frac{ax}{a^2x^2-b^2}\sqrt{\frac{b^2y}{a}}\times\frac{ax-b}{bx}\sqrt[4]{cx^3}$.

14. $5b^3x\sqrt{2xy}\times 2cy\sqrt[5]{x^3y^2}$.

15. $(a-x)\sqrt[4]{3x^3}\times(a+x)\sqrt[3]{2x^2}\times(a^2+x^2)\sqrt{\frac{1}{4x^6}}$.

16. $2xy^2\sqrt[m]{a}\times 3x^2y\sqrt[n]{xy}$.

17. $\sqrt[4]{\left(\frac{16}{25}\right)^7}\times\sqrt[4]{\left(\frac{25}{64}\right)^6}$.

18. $\sqrt[3]{(4ab^2)^x}\times\sqrt[3]{(2a^2b)^x}$.

19. $\sqrt[2n]{x^3}\times\sqrt[n]{x}\times\sqrt[3n]{x^2}$.

20. $\frac{a}{y}\sqrt{\frac{b}{x}}\times\frac{y}{c}\sqrt[3]{\frac{a^2}{y^2}}\times c\sqrt{\frac{ac^2}{y^3}}$.

21. $\sqrt{\frac{ax^2}{(a+x)^3}}\times\sqrt{\frac{b(a^2-x^2)^3}{x^5}}\times\sqrt[4]{\frac{a^3y^3}{(a-x)^6}}$.

MULTIPLICATION OF POLYNOMIAL RADICALS.

By combining the preceding rules with that for the multiplication of polynomials, we may multiply together radical expressions consisting of any number of terms.

EXAMPLES.

1. Multiply $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

2. Multiply $a^{\frac{3}{4}}+2a^{\frac{1}{2}}-a^{\frac{1}{4}}$ by $a^{\frac{1}{2}}-3a^{\frac{1}{4}}+2$.

3. Multiply $a^{\frac{7}{2}}-a^3+a^{\frac{5}{2}}-a^2+a^{\frac{3}{2}}-a+a^{\frac{1}{2}}-1$ by $a^{\frac{1}{2}}+1$.

4. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$.

5. Multiply $\sqrt{2}-\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$.

6. Multiply $3\sqrt{5}+2\sqrt{3}$ by $3\sqrt{5}+2\sqrt{3}$.

7. Multiply $\sqrt{x+1}-\sqrt{x-1}$ by $\sqrt{x+1}+\sqrt{x-1}$.

8. Multiply $\sqrt[3]{12+\sqrt{19}}$ by $\sqrt[3]{12-\sqrt{19}}$.

9. Multiply $\sqrt{a}+c\sqrt[3]{b}$ by $\sqrt{a}-c\sqrt[3]{b}$.

10. Multiply $x^2-x\sqrt{2}+1$ by $x^2+x\sqrt{2}+1$.

11. Multiply $5+\sqrt{2}+2\sqrt[4]{2}$ by $\sqrt{2}+\sqrt[3]{2}$.

DIVISION OF SURDS.

233. The principles employed in the division of monomial surds are the same as those used in the division of rational monomials.

Let it be required to give:

1. The *first rule* for the division of the factors of monomial quantities.

2. The *second rule* for the division of the factors of monomial quantities.

Monomial surds can always be reduced to equivalent expressions with the same fractional exponents, or, in the radical form, to a common index. The *second rule* for the division of the factors of monomials is therefore always applicable.

Care should be taken to reduce the quotient to its simplest form. In radicals of the second and third degree, the quantity under the radical should not be left in a fractional form. In radicals of a higher degree, the fractional form under the radical is generally retained.

Illustrative Example.

Find the quotient of $(2ab)^{\frac{1}{2}} \div (4a^2b)^{\frac{1}{3}}$.

$$(2ab)^{\frac{1}{2}} = (2ab)^{\frac{3}{6}} = [(2ab)^3]^{\frac{1}{6}} = (8a^3b^3)^{\frac{1}{6}}.$$

$$(4a^2b)^{\frac{1}{3}} = (4a^2b)^{\frac{2}{6}} = [(4a^2b)^2]^{\frac{1}{6}} = (16a^4b^2)^{\frac{1}{6}}.$$

$$\text{Quotient} = \left(\frac{b}{2a}\right)^{\frac{1}{6}}.$$

The foregoing example, written in the radical form, would read as follows:

Find the quotient of $\sqrt{2ab} \div \sqrt[3]{4a^2b}$.

$$\sqrt{2ab} = \sqrt[6]{8a^3b^3}$$

$$\sqrt[3]{4a^2b} = \sqrt[6]{16a^4b^2}$$

$$\text{Quotient} = \sqrt[6]{\frac{b}{2a}}.$$

EXAMPLES.

Divide the following:

1. $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{5}}$.

2. $4(ab)^{\frac{1}{2}}$ by $2(ab)^{\frac{1}{3}}$.

3. $5a(x^2y)^{\frac{1}{3}}$ by $a(xy^2)^{\frac{1}{5}}$.

4. $4^{\frac{1}{2}}$ by $2^{\frac{3}{2}}$.

5. $(16a^3 - 12a^2x)^{\frac{1}{2}}$ by $2a$.

6. $bc(a^3x^2)^{\frac{1}{n}}$ by $c(ax)^{\frac{1}{n}}$.

7. $(a^3c)^{\frac{1}{m}}$ by $(ac^3)^{\frac{1}{n}}$.

8. $8\sqrt{108}$ by $2\sqrt{6}$.

9. $15a^5x^3\sqrt[3]{54a^2}$ by $3a^2x\sqrt[5]{2a}$.

10. $4x^2m^3\sqrt[3]{12}$ by $2x^m m^2\sqrt{3}$.

11. $12a^3x^2\sqrt[5]{4a^3b^4}$ by $2\sqrt[3]{a^2b}$.

12. $7a^2b\sqrt{abc}$ by $ab\sqrt[5]{ab^2c^3}$.

13. $\sqrt[4]{x^2z - xz^2}$ by $\sqrt{xz}$.

14. $\sqrt[3]{\dfrac{my}{2nz^2}}$ by $\sqrt[6]{\dfrac{a}{16nyz}}$

15. $\dfrac{\sqrt{a}}{\sqrt[3]{x}}$ by $\dfrac{\sqrt[4]{a^3}}{\sqrt[3]{x^2}}$

By combining the preceding rules with that for the division of polynomials, we may divide one radical expression by another when each contains any number of terms.

EXAMPLES.

1. Divide $a^{\frac{4}{5}} - 6a^{\frac{2}{5}} + 25$ by $a^{\frac{2}{5}} - 4a^{\frac{1}{5}} + 5$.

2. Divide $a^{\frac{1}{2}} + 64b^{\frac{1}{2}}$ by $a^{\frac{1}{6}} + 4b^{\frac{1}{6}}$.

3. Divide $a^{\frac{4}{3}}-8a^{\frac{1}{3}}b$ by $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+4b^{\frac{2}{3}}$.

4. Divide $3\sqrt{5}+\sqrt{72}-\sqrt{8}$ by $\sqrt{2}$.

5. Divide $8\sqrt{2}-16\sqrt[3]{2}+4\sqrt[4]{2}$ by $2\sqrt[6]{2}$.

6. Divide $2\sqrt{ac}-2\sqrt{cy}+3\sqrt{ax}-3\sqrt{xy}$ by $\sqrt{a}-\sqrt{y}$.

7. Divide $2+\sqrt[6]{32}-\sqrt[4]{8}-\sqrt[12]{128}$ by $\sqrt{2}+\sqrt[3]{2}$.

8. Divide $6+3\sqrt{3}-2\sqrt[3]{2}-\sqrt[6]{108}$ by $3-\sqrt[3]{2}$.

9. Divide $x+\sqrt{xy}+y$ by $\sqrt{x}+\sqrt[4]{xy}+\sqrt{y}$.

10. Divide $\dfrac{x-7\sqrt{x}}{x-5\sqrt{x}-14}$ by $\dfrac{\sqrt{x}}{\sqrt{x}+2}$.

11. Divide $\dfrac{\sqrt{x}-\sqrt[3]{x}}{(x-\sqrt[3]{x^2})\sqrt{x}}$ by $\dfrac{\sqrt{x}}{\sqrt{x}+\sqrt[3]{x}}$.

INVOLUTION OF SURDS.

234. The involution of monomial surds gives rise to two cases:

I. *When the surd is entire.*

II. *When the surd is mixed.*

Case I. The different methods of raising an entire surd to any power have been already explained in article 224.

Case II. The rational and surd factors must each be raised to the required power.

The final result should be written in its simplest form.

EXAMPLES.

1. Required the fourth power of $\frac{2}{3}a^{\frac{1}{3}}$.

2. Required the sixth power of $(a+b)^{\frac{1}{5}}$.

3. Required the second power of $3(x-y)^{\frac{1}{2}}(a+b)^{\frac{3}{4}}$.

4. Required the second power of $3\sqrt[3]{3}$.

5. Required the fourth power of $\frac{1}{6}a\sqrt[3]{6a^2}$.

6. Required the fourth power of $2ax\sqrt[6]{3a^4b}$.

7. Required the third power of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

8. Required the fifth power of $\frac{bx}{ay}\sqrt[4]{\frac{bx}{ay}}$.

9. Required the second power of $3ab\sqrt[4]{2ac^2}$.

10. Required the eighth power of

$$(x-z)\sqrt[12]{y^5(x-z)^2}.$$

11. Required the second power of $3-\sqrt{2}$.

12. Required the third power of $\sqrt{2}-\sqrt{3}$.

13. Required the second power of

$$\sqrt{\frac{1+x}{2}}+\sqrt{\frac{1-x}{2}}.$$

EVOLUTION OF SURDS.

235. As in the evolution of rational quantities, the root to be extracted may be indicated by a fractional exponent or the radical sign. This will, in general, involve two fractional exponents or two radicals in the same expression. A translation of the first fractional exponent or first radical into common language, often serves to bring more clearly before the mind the work required.

Translate into common language the first fractional exponent or first radical, in each of the following:

1. $(x^{\frac{1}{3}} y^{\frac{1}{3}})^{\frac{1}{4}}$.
2. $[(a-b)^{\frac{1}{3}}]^{\frac{1}{2}}$.
3. $\sqrt{\sqrt{2ax}}$.
4. $\sqrt[3]{\sqrt{2a-x}}$.
5. $\sqrt{\sqrt[3]{(2ax-y)}}$.
6. $\sqrt[4]{2ab\sqrt{3xy}}$.

In extracting the root of surds two cases arise:

I. *When the surd is entire.*

II. *When the surd is mixed.*

Case I. This case has been already explained in article 224.

Case II. The extraction of any root of a mixed surd involves evolution, multiplication, and reduction. A change in the order of these steps will not affect the final result. In general, it is found most convenient to proceed under the following

RULE.

Extract the root of the rational factor if possible, use it as the coefficient of the root of the surd factor and reduce the result to its simplest form. If the exact root of the rational factor cannot be obtained, indicate the required root of this *factor, multiply the result by the root of the surd factor and simplify the product; or, reduce the mixed surd to an entire surd, extract the required root, and write the result in its simplest form.*

EXAMPLES.

1. Find the square root of $9(3)^{\frac{1}{3}}$.

2. Find the fourth root of $\frac{16}{81}a^{\frac{2}{3}}$.

3. Find the mth root of $5(a+b)^{\frac{1}{2}}$

4. Find the third root of $(6x^2 - 2x^3)^{\frac{3}{2}}$.

5. Find the sixth root of $\left(\frac{x^3y^6}{z^6a^3}\right)^{\frac{2}{7}}$.

6. Find the cube root of $\frac{1}{8}\sqrt{2}$.

7. Find the cube root of $64a^3x^9\sqrt[4]{27x^6}$.

8. Find the sixth root of $m^3n^6\sqrt{27n^9}$.

9. Find the cube root of $\frac{x}{2}\sqrt{\frac{x}{2}}$.

10. Find the square root of $3x^2\sqrt[3]{108}$.

11. Find the fourth root of $\frac{4}{25}\sqrt[3]{\frac{4}{25}}$.

12. Find the sixth root of $729a^{12}\sqrt[4]{8x^3}$.

13. Find the 18th root of $\sqrt[5]{a^{90}b^9}$.

IMAGINARY QUANTITIES.

236. Algebraic quantities are broadly divided into two great classes, real quantities and imaginaries.

237. A quantity which can be conceived as positive or negative is a **real quantity.**

Thus,

$$a,\ a^n,\ \sqrt[3]{a} \text{ and } \sqrt[4]{a^3} \text{ are real quantities.}$$

238. Quantities which cannot be conceived as positive or negative are **imaginary.**

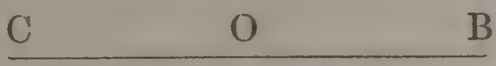
Thus,

$$\sqrt{-4} \text{ is imaginary; } -4 \text{ is real.}$$

239. Every algebraic quantity conveys the two ideas of numerical magnitude and quality. The former may be represented by the length of a line; the latter by its direction. Any direction may be taken as positive; the opposite will be negative. Lines representing *real* quantities must lie in one or the other of these directions; quantities represented by lines lying in any other direction are imaginary.

COMPLETE REVERSALS.

240. Assuming distances measured to the right as positive, let us represent any quantity, a, by the line OB.



We can conceive this line to be revolved about the point O, either in a direction contrary to the hands of a clock or in the same direction. Custom has fixed upon the former as *positive* revolution. If the line OB is revolved until it takes the position OC, this will constitute a *complete reversal* of direction. It evidently amounts to multiplying a by -1, and gives us the negative line, $-a$. A second reversal will complete the revolution or multiply $-a$ by -1 and give us the original line, a.

Therefore,

Multiplication by $(-1)^0$ signifies no reversal.
Multiplication by (-1) signifies one reversal.
Multiplication by $(-1)^2$ signifies two reversals.
Multiplication by $(-1)^3$ signifies three reversals,
and so on.

It is plain that multiplication by $(-a)^0$, $(-a)$, $(-a)^2$, $(-a)^3$, will effect the same reversals and in addition multiply the length of the line by a^0, a, a^2 and a^3 respectively. In all cases, the complete reversal of a line representing a real quantity, positive or negative, will give a line representative of a real quantity, but with a contrary sign.

PARTIAL REVERSALS.

241. Multiplication of a line by -1 reverses its direction; since the $\sqrt{-1}$ squared equals -1, multiplication by the $\sqrt{-1}$ must necessarily revolve the line to such position that a second multiplication by the $\sqrt{-1}$ will completely reverse the line. Multiplication by the $\sqrt{-1}$, of lines representing real quantities, either rational or surd, must consequently give lines perpendicular to the direction of real lines. Considering real quantities to be represented by horizontal lines, $4\sqrt{-1}$, for instance, will be represented by a vertical line 4 units in length, extending upward, and $-4\sqrt{-1}$, by a vertical line of the same length, extending downward. Since all imaginary quantities of the second degree can be factored into a real quantity multiplied by $\sqrt{-1}$, each such imaginary can be represented by a vertical line.

CONSTRUCTIONS.

Construct lines representing the following imaginary quantities:

1. $3\sqrt{-1}$, $a\sqrt{-1}$, $-5\sqrt{-1}$.

2. $7\sqrt{-1}$, $\sqrt{-9}$, $\sqrt{-5}$.

3. $\sqrt{-a^2}$, $\sqrt{-a}$, $\sqrt{-a^3}$.

4. $-\sqrt{-4}$, $-\sqrt{-9a}$, $a\sqrt{-a}$.

ADDITION AND SUBTRACTION OF IMAGINARY QUANTITIES.

242. It is evident that lines that are neither in the same nor opposite directions cannot be united into one line; neither can the quantities they represent be united into one term. All imaginary quantities of the second degree are, however, represented by lines lying in the same or opposite directions. They can, therefore, be added and subtracted, the same as real quantities.

EXAMPLES.

1. Simplify $ab - \sqrt{-a} + 2\sqrt{-a} + 3ab$.

2. Subtract $2a + 3\sqrt{-b}$ from $7a - 5\sqrt{-b}$.

3. Subtract $8\sqrt{-1}$ from $-12\sqrt{-1}$.

4. Simplify $\sqrt{-9} + \sqrt{-16} + 2\sqrt{-25}$.

5. Simplify $2\sqrt{-81a^4} + \sqrt{-4b^2} + \sqrt{-36c^6}$.

6. Simplify $2\sqrt{-8a^3b} + 3a\sqrt{-18ab^3}$.

7. Simplify $4x\sqrt{-a^3-a^2b} + 2a\sqrt{-ax^2-bx^2}$.

MULTIPLICATION OF IMAGINARY QUANTITIES.

243. Two cases arise:

Case I. *When an imaginary and a real quantity are multiplied together.*

In this case it is clear that the line representing the imaginary quantity is changed in length, but that its direction either remains unchanged or is entirely reversed. This is the same as in the multiplication of real quantities. The process of multiplication is therefore similar.

Hence the

RULE.

Multiply as in the multiplication of real quantities.

EXAMPLES.

1. Multiply $\sqrt{-1}$ by 3.

2. Multiply $\sqrt{18}$ by $\sqrt{-2}$.

3. Multiply $\sqrt{-2a}$ by $\sqrt{8a}$.

4. Multiply $a+\sqrt{-b}$ by $\sqrt{b}$.

5. Multiply $3ax\sqrt{-4}$ by $6a^2y\sqrt{36}$.

6. Multiply $(a-x)\sqrt{-3b}$ by $(a+x)\sqrt{3b}$.

Case II. *When one imaginary quantity is multiplied by another.*

In this case the multiplicand and multiplier can each be separated into a real factor multiplied by $\sqrt{-1}$. The multiplicand signifies that the line representing its real factor has been subjected to one-half of a complete reversal. Multiplication by the given multiplier, evidently multiplies the length of the line by the real factor of this multiplier, and subjects the resultant line to another half reversal. The final line, then, is evidently the line representing the product of the two real factors, subject to one entire reversal, or in other words, multiplied by -1.

Illustrative Examples.

Multiply $\sqrt{-3}$ by $\sqrt{-2}$.

$$\sqrt{-3} = \sqrt{3}\sqrt{-1}$$
$$\underline{\sqrt{-2} = \sqrt{2}\sqrt{-1}}$$
$$\sqrt{6} \times -1 = -\sqrt{6}.$$

Draw a line to represent the multiplicand in the above example, and show what resultant line is obtained by multiplication.

Multiply $\sqrt{-3a}$ by $\sqrt{-2b}$.

$$\sqrt{-3a} = \sqrt{3a}\sqrt{-1}$$
$$\underline{\sqrt{-2b} = \sqrt{2b}\sqrt{-1}}$$
$$\sqrt{6ab} \times -1 = -\sqrt{6ab}.$$

No multiplication of imaginary factors is correct that does not take account of the reversal involved in such multiplication, or, in other words, multiply the product of the real factors by — 1.

In multiplying together either two real factors or two imaginary factors, the double sign of the product is not generally used. If, for any reason, the double sign is employed, in a particular example, it must be noticed that if the quantities multiplied together are imaginary, the double sign arising from the multiplication of the real factors in these quantities is subject to reversal, or multiplication by — 1.

Thus, $\sqrt{12} \times \sqrt{3}$ gives $\pm \sqrt{36}$

$$\sqrt{-12} \times \sqrt{-3} \text{ gives } \pm \sqrt{36} \times -1 = \mp \sqrt{36}.$$

RULE.

Separate the multiplicand and multiplier each into a real factor multiplied by $\sqrt{-1}$; *multiply the factors together, remembering that* $\sqrt{-1} \times \sqrt{-1} = -1$; *then write the result in its simplest form.*

EXAMPLES.

1. Multiply $\sqrt{-3}$ by $\sqrt{-5}$

2. Multiply $\sqrt{-9}$ by $\sqrt{-16}$.

3. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$.

4. Multiply $3\sqrt{-5}$ by $-2\sqrt{-2}$.

5. Multiply together $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, and $\sqrt{-5}$.

6. Multiply $-\sqrt{-1}$ by $\sqrt{-1}$.

7. Multiply $2a\sqrt{-3}$ by $5b\sqrt{-a}$.

8. Multiply $\frac{3(x+y)\sqrt{-x}}{2}$ by $\frac{4\sqrt{-x}}{x^2-y^2}$.

9. Multiply $\frac{3\sqrt{-x-y}}{\sqrt{a^2-b^2}}$ by $\frac{\sqrt{a+b}\sqrt{-x-y}}{6\,(x+y)}$.

By combining the preceding rule with that for the multiplication of polynomials, we may multiply together radical expressions containing any number of imaginary terms.

10. Multiply $a+\sqrt{-b}$ by $a-\sqrt{-b}$.

11. Multiply $2\sqrt{-3}-3\sqrt{-2}$ by $4\sqrt{-3}+6\sqrt{-2}$.

12. Multiply $\sqrt{5}+2\sqrt{-3}$ by $\sqrt{2}-\sqrt{-3}$.

13. Multiply $\sqrt{a}+\sqrt{-c}$ by $\sqrt{-a}+\sqrt{c}$.

14. Multiply $\sqrt{-3}+\sqrt{-4}+\sqrt{-5}$ by $\sqrt{-3}$.

15. Multiply $2\sqrt{-2}+3\sqrt{-3}+\sqrt{-5}$ by $3\sqrt{-2}$
$-4\sqrt{-3}$.

16. Find the square of $\sqrt{-2}+\sqrt{-3}$.

17. Find the square of $2\sqrt{-3}-3\sqrt{-4}$.

18. Find the cube $\frac{1+\sqrt{-3}}{2}$.

19. Find the cube of $\frac{-1-\sqrt{-3}}{2}$.

20. Find the cube of $\frac{-3+3\sqrt{-3}}{2}$.

21. Find the cube of $\frac{-3-3\sqrt{-3}}{2}$.

22. Find the cube of $1-\sqrt{-3}$.

DIVISION OF IMAGINARY QUANTITIES.

244. Two cases arise corresponding to the two cases in multiplication.

Case I. In case I in multiplication, two factors, one a *real* and the other an *imaginary* quantity, were multiplied together, giving an imaginary quantity as the product. In the corresponding case in division, *the product or dividend, which is imaginary, is divided by one of these factors*, giving the other as a quotient. Since the process of multiplication in case I is the same as in the multiplication or real quantities, it follows that the converse steps in division will be the same as in the division of real quantities.

Hence the

RULE.

Divide as in the division of real quantities.

EXAMPLES.

1. Divide $3\sqrt{-6}$ by $\sqrt{3}$.
2. Divide $4\sqrt{-8}$ by $2\sqrt{-2}$.
3. Divide $7\sqrt{-4}+4\sqrt{-10}$ by $2\sqrt{-2}$.
4. Divide $8\sqrt{-8}+4\sqrt{2}-6\sqrt{-3}$ by $2\sqrt{3}$.
5. Divide $x^2\sqrt{-ab}+ax\sqrt{-bc}$ by $x\sqrt{b}$.
6. Divide $am^3\sqrt{-cx}-m^2n^2\sqrt{-cyz}$ by $m^2\sqrt{-c}$.

Case II. In case II in multiplication, two imaginary quantities multiplied together produced a real quantity as the product. In the corresponding case in division, *the product or dividend which is real is divided by an imaginary quantity*, giving an imaginary quantity as the quotient. In the case in multiplication, the imaginary factors of the multiplicand and multiplier ($\sqrt{-1}$ and $\sqrt{-1}$) multiplied together produce the factor -1 in the product. In division this factor, -1, in the product or dividend must be separated into $\sqrt{-1} \times \sqrt{-1}$. The divisor should also be separated into a real factor multiplied by $\sqrt{-1}$.

All factors common to dividend and divisor (or numerator and denominator, if the division is expressed in fractional form) should be discarded. The result will be the quotient desired.

Hence the

RULE.

Express the division in the form of a fraction; separate the numerator into two factors, one of which shall be -1 *and replace* -1 *by its factors,* $\sqrt{-1} \times \sqrt{-1}$*; separate the denominator into a real factor* $\times \sqrt{-1}$*; strike out all factors common to the numerator and denominator and write the result in its simplest form.*

EXAMPLES.

1. Divide $\sqrt{6}$ by $\sqrt{-2}$.

$$\frac{\sqrt{6}}{\sqrt{-2}} = \frac{-\sqrt{6} \times -1.}{\sqrt{-2}} = \frac{-\sqrt{6}\sqrt{-1}\sqrt{-1}}{\sqrt{2}\sqrt{-1}} =$$

$$-\sqrt{3}\sqrt{-1} = -\sqrt{-3}.$$

2. Divide 1 by $\sqrt{-1}$. 3. Divide a by $\sqrt{-a}$.

4. Divide $\sqrt{84}$ by $\sqrt{-7}$.

5. Divide 2 by $\sqrt{-3}$.

6. Divide $-\sqrt{25}$ by $\sqrt{-5}$.

7. Divide x by $\sqrt{-x^2}$.

8. Divide a by $b\sqrt{-1}$.

9. Divide $2\sqrt{8}-\sqrt{-10}$ by $-\sqrt{-2}$.

10. Divide $5-\sqrt{-3}$ by $\sqrt{-3}$.

11. Divide $\sqrt{15}+\sqrt{5}-\sqrt{-10}$ by $\sqrt{-5}$.

12. Divide $9a^2b\sqrt{12}+3ax^3\sqrt{-4}-6a^3b\sqrt{6}$
by $3a\sqrt{-2}$.

13. Divide $3a\sqrt{x^2-y^2}+24b(x+y)$
by $3\sqrt{-x-y}$.

RATIONALIZING FRACTIONS.

245. Any algebraic expression containing a surd is called an irrational or surd quantity.

An irrational numerator or denominator can be rationalized and the value of the fraction not changed, by multiplying both terms of the fraction by the proper multiplier. Since an irrational divisor is, in general, more troublesome than an irrational dividend, it is usually the denominator which is rationalized.

Three cases will be considered:

246. Case 1. *When the denominator is in monomial form.*

RULE.

Multiply both numerator and denominator by some quantity that will render the denominator rational.

A convenient method of determining this multiplier is to express the denominator in the form of a quantity affected by a fractional exponent. The required multiplier will evidently be this same quantity affected by an exponent, which added to the exponent of the denominator, will equal the integer next in value above that fractional exponent. This multiplier can then be expressed in the radical form. Let the denominator to be rationalized be $\sqrt[3]{b^2}$. $\sqrt[3]{b^2} = b^{\frac{2}{3}}$; the multiplier is evidently $b^{\frac{1}{3}}$ and the rational product b. In radical form the multiplier is $\sqrt[3]{b}$.

EXAMPLES.

Rationalize the denominators in the following:

1. $\dfrac{2}{\sqrt{3}}$
2. $\dfrac{a}{b\sqrt{c}}$
3. $\dfrac{a}{3\sqrt{2b}}$
4. $\dfrac{a^2}{\sqrt[3]{b^2}}$
5. $\dfrac{x}{\sqrt[4]{y^7}}$
6. $\dfrac{xz^2}{\sqrt[3]{2ax}}$
7. $\dfrac{a-b}{\sqrt{a+b}}$
8. $\dfrac{2}{\sqrt[3]{(x-y)^2}}$
9. $\dfrac{3ax}{\sqrt[n]{b}}$
10. $\dfrac{4}{\sqrt[n]{c^3}}$
11. $\dfrac{2}{\sqrt[5]{16x^3}}$
12. $\dfrac{2\sqrt{2}}{3\sqrt[3]{4}}$
13. $\dfrac{4+\sqrt{3}}{2\sqrt{3}}$
14. $\dfrac{5-\sqrt{-3}}{\sqrt{-3}}$
15. $\dfrac{1-\sqrt{2}}{\sqrt[3]{3}}$
16. $\dfrac{a-\sqrt[n]{b}}{\sqrt[n]{b^{n-1}}}$

247. Case II. *When the denominator can be put in the form of a binomial containing surds of the second degree only.*

RULE.

Multiply both terms of the fraction by the denominator with the sign between the terms of the binomial changed.

When the denominator is composed of several monomial terms, care should be exercised to put it into such binomial form that the resultant denominator, obtained by a single multiplication, will contain as few radical terms as possible. For instance, when the radical appears in only one term, this term should constitute one of the terms of the binomial. Repeated multiplications, however, are necessary to rationalize a denominator containing several monomial radicals.

EXAMPLES.

Rationalize the denominators in the following:

1. $$\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}\,(3+\sqrt{2})}{(3-\sqrt{2})\,(3+\sqrt{2})}=\frac{3\sqrt{2}+2}{9-2}=\frac{3\sqrt{2}+2}{7}\text{ Ans.}$$

2. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

3. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$

4. $\dfrac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}$

5. $\dfrac{\sqrt{3}+\sqrt{-2}}{\sqrt{3}-\sqrt{-2}}$

6. $\dfrac{\sqrt{-5}+\sqrt{-3}}{\sqrt{-4}-\sqrt{-6}}$

7. $\dfrac{1+x+\sqrt{1-x^2}}{1+x-\sqrt{1-x^2}}$

8. $\dfrac{1}{\sqrt{2}+\sqrt{3}+1}$

9. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

248. Case III. *When the denominator is a simple binomial containing radicals of a higher degree than the second.*

RULE.

Write each term of the denominator in the fractional exponent form and then raise these terms to the power indicated by the lowest common multiple of the denominators of these fractional exponents. Connect the results by such sign as will render the binomial thus formed, exactly divisible by the denominator (Art. 90); perform such division, and multiply both terms of the fraction by the quotient.

EXAMPLES.

Reduce each of the following to an equivalent fraction having a rational denominator.

1. $\dfrac{\sqrt{a}+\sqrt[3]{b}}{\sqrt{a}-\sqrt[3]{b}}$.

In this example the multiplier is $\dfrac{(a^{\frac{1}{2}})^6-(b^{\frac{1}{3}})^6}{a^{\frac{1}{2}}-b^{\frac{1}{3}}}$

$$=\frac{a^3-b^2}{a^{\frac{1}{2}}-b^{\frac{1}{3}}}=a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}.$$

2. $\dfrac{1}{\sqrt[3]{x}+\sqrt[6]{y}}$.

3. $\dfrac{a^{\frac{1}{2}}b^{\frac{2}{3}}}{a^{\frac{1}{2}}-b^{\frac{2}{3}}}$.

4. $\dfrac{\sqrt{2}+\sqrt[3]{3}}{\sqrt{2}-\sqrt[3]{3}}$.

5. $\dfrac{1}{\sqrt{a}+\sqrt[5]{b}}$.

249. The utility of rationalizing the denominator is apparent when we attempt to compute the numerical value of a fractional surd.

EXAMPLES.

Compute the value of each of the following to four decimal places:

1. $\dfrac{3}{\sqrt{2}-1}$.

2. $\dfrac{2}{\sqrt[3]{3}}$.

3. $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$.

4. $\dfrac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$.

SQUARE ROOT OF A BINOMIAL SURD.

250. A **quadratic surd** is a monomial surd of the second degree.

251. A **binomial surd** is a binomial, one or both of whose terms are quadratic surds.

Thus, $2+\sqrt{3}$ and $\sqrt{3}-\sqrt{7}$ are binomial surds.

252. The square of a binomial surd, of either form, is equal to a rational quantity plus or minus a surd.

Thus, $(2+\sqrt{3})^2 = 4+4\sqrt{3}+3 = 7+4\sqrt{3}$, a rational quantity + a surd.

$$(\sqrt{3}-\sqrt{7})^2 = 3-2\sqrt{21}+7 = 10-2\sqrt{21},$$

a rational quantity — a surd.

It does not follow that every rational quantity, plus or minus a surd, is such a square. Representing, however, the rational term of the general example by a and the surd term by $\sqrt{b}$, such square can be expressed by $a \pm \sqrt{b}$, in which a and b represent only those values which render the expression as a whole the square of a binomial surd. It is plain that the square root of such a square will be a binomial surd containing a rational, and a surd term, or having both terms surds. As the rational term can be written in the radical form, the square root can, in either case, be written in the form $\sqrt{x} \pm \sqrt{y}$, in which, when the values of x and y are found, the required square root will be known.

Evidently, in squaring a binomial surd of either form, the square of the sum of any two terms will be the same as the square of the difference of the same two terms, except that in the first case, the sign before the surd term of the square will be plus, and in the second case minus. It follows that the sign between the two terms of the root will be plus or minus as the sign before the surd term in the square is plus or minus; and that, whenever we have $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then we will also have $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$, where a, b, x and y have the same values in the two equations.

Summarizing these results, we have the following

PRINCIPLES.

1. *The square of any binomial surd can be represented by* $a \pm \sqrt{b}$; *in which* a *and* b *have only such values as render the expression the square of a binomial surd.*

2. *The square root of such a square is equal to* $\sqrt{x} \pm \sqrt{y}$, *in which* x *and* y *represent unknown quantities, whose values are to be found in each example.*

3. *The sign between the two terms of the root will be plus or minus as the sign before the surd term of the square is plus or minus.*

These principles will enable us to find the square root of the square of any binomial surd.

Let us first take the general example $a \pm \sqrt{b}$.

We have (1) $\sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y}$,

or separating into two equations, (2) $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

(3) $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Regarding (2) and (3) as two equations containing the two unknown quantities, x and y, we proceed to their solution.

Eliminating $\sqrt{y}$, (4) $\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}} = 2\sqrt{x}$.

Squaring both members, (5) $a + \sqrt{b} + 2\sqrt{a^2 - b} + a - \sqrt{b} = 4x$.

Collecting terms, (6) $2a + 2\sqrt{a^2 - b} = 4x$.

Solving for x, (7) $x = \dfrac{a + \sqrt{a^2 - b}}{2}$.

$$\sqrt{x} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}}.$$

Eliminating $\sqrt{x}$ in equations (2) and (3) we can, in a similar way obtain

$$y = \frac{a - \sqrt{a^2 - b}}{2}.$$

$$\sqrt{y} = \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

Substituting these values of $\sqrt{x}$ and $\sqrt{y}$ in equation (1) we have,

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}},$$

or, separating the two answers and denoting the first by A and the second by B, we have,

$$(A)\quad \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}},$$

$$(B)\quad \sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

Any particular example can be solved by substituting in the answer to the corresponding general example, the values of a and b given in the particular example. Care must be taken, in determining b, to put the coefficient of the surd term in the square, under the radical sign, before comparing the particular with the general example.

EXAMPLES.

1. Required to find the square root of $4+2\sqrt{3}$.

Put in the proper form for comparison, this equals $4+\sqrt{12}$.

Comparing this with the general example, $a+\sqrt{b}$, $a=4$, $b=12$.

Taking formula (A),

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

and substituting in this formula the values of a and b for this particular example we have

$$\begin{aligned}\sqrt{4+2\sqrt{3}} &= \sqrt{4+\sqrt{12}}\\ &=\sqrt{\frac{4+\sqrt{16-12}}{2}}+\sqrt{\frac{4-\sqrt{16-12}}{2}}\\ &=\sqrt{3}+1.\end{aligned}$$

This answer may be tested thus, $(\sqrt{3} + 1)^2 = 3 + 2\sqrt{3} + 1 = 4 + 2\sqrt{3}$.

Find the square root of the following:

2. $33 - 20\sqrt{2}$.
3. $67 + 7\sqrt{72}$.
4. $\frac{3}{2} + \sqrt{2}$.
5. $42 - 3\sqrt{174\frac{2}{9}}$.
6. $(a + b)^2 - 4(a - b)\sqrt{ab}$.
7. $1 - 4\sqrt{-3}$.
8. $-2\sqrt{-1}$.
9. $\frac{a^2}{4} + \frac{c}{2}\sqrt{a^2 - c^2}$.
10. $\frac{3a}{b} + \sqrt{\frac{12a^3c^2}{bd^2} - \frac{4a^4c^4}{d^4}}$.
11. $b^2 - ab + \frac{a^2}{4} + \sqrt{4ab^3 - 8a^2b^2 + a^3b}$.

253. The square root of a binomial surd, when such binomial is a perfect square, can also be found by inspection.

The binomial is first reduced to a trinomial with its terms so arranged that the second term is twice the square root of the first and third. The square root of this expression is evidently the square root of its first and third terms connected by the sign preceding the second term of the trinomial.

This process, stated in the form of a rule, will be as follows:

RULE.

1. *Write the surd term so it will have the coefficient 2.*

2. *Separate the rational term into two terms whose product shall equal the quantity under the radical sign of the surd term in its new form.*

3. *Extract the square root of each of these terms and connect the results by the sign of the surd term.*

EXAMPLES.

Find the square root of the following:

1. $14+6\sqrt{5}=14+2\sqrt{45}=9+2\sqrt{45}+5$.

Ans. $3+\sqrt{5}$.

2. $7+2\sqrt{10}$.
3. $10+2\sqrt{21}$.
4. $14-4\sqrt{6}$.
5. $8-\sqrt{60}$.
6. $20+3\sqrt{44}$.
7. $2m+2\sqrt{m^2-n^2}$.
8. $2a+x+2\sqrt{a^2+ax}$.
9. $-2-2\sqrt{-15}$.

SOLUTION OF RADICAL EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

254. The steps to be taken are in general similar to those in the solution of equations composed of rational quantities, except that there is the additional work of rationalizing the terms containing the unknown quantity.

Remarks on Examples.

1. Whenever an indicated operation, such as division, multiplication, etc., can be performed to advantage, perform it.

2. It is sometimes helpful to rationalize the denominator of a fractional term.

3. Care should be taken, during the progress of the work, to cancel all factors common to the members of the equation.

4. If the equation contains but one radical term, place that by itself on one side of the equation and involve both members to a power denoted by the index of the radical.

5. If the equation contains two or more radical terms, successive involutions will, in general, be necessary. After each involution, the terms should be collected and, in general, a single radical term (usually the most complicated one) placed by itself in one member, before the next involution is performed.

6. In many examples the method of procedure and the best arrangement of terms can be determined only by careful inspection and trial.

Illustrative Example.

Solve $3-\sqrt{21+x-\sqrt{2+x}}=\sqrt{2+x}$.

Placing the more complicated radical in one member, we have

$$3-\sqrt{2+x}=\sqrt{21+x-\sqrt{2+x}}.$$

Squaring both members,

$$9-6\sqrt{2+x}+2+x=21+x-\sqrt{2+x}.$$

Collecting terms, $-5\sqrt{2+x}=10.$

Dividing by -5, $\sqrt{2+x}=-2.$

Squaring again $2+x=4$

$$x=2.$$

EXAMPLES.

Solve the following equations:

1. $\sqrt{2x-4}-\sqrt{2x+1}=-1.$

2. $\sqrt{5x}+\sqrt{5x-9}=\dfrac{36}{\sqrt{5x-9}}$.

3. $\sqrt[3]{3x+6}+4=7$.

4. $\sqrt{x^2+5x+7}+\sqrt{x^2-7x-5}=6$.

5. $m+x=\sqrt{m^2+x\sqrt{n^2+x^2}}$.

6. $\sqrt{x-24}=12-\sqrt{x}$.

7. $(x+3)^{\frac{1}{m}}=(x^2-10x+17)^{\frac{1}{2m}}$.

8. $\sqrt{7+\sqrt{1+\sqrt{3x}}}=3$.

9. $\sqrt{3a^2-\sqrt{3a^4+\sqrt{3a^8-\sqrt{2a^{16}x}}}}=a$.

10. $\dfrac{x-m}{\sqrt{x}+\sqrt{m}}-2\sqrt{m}=\dfrac{\sqrt{x}-\sqrt{m}}{3}$.

11. $\sqrt{\dfrac{2}{4+x}}+\sqrt{\dfrac{3}{4-x}}=\sqrt[4]{\dfrac{24}{16-x^2}}$.

12. $\dfrac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}-\sqrt{x}}=c$.

13. $\sqrt{\sqrt{x}+a}-\sqrt{\sqrt{x}-a}=\sqrt{2\sqrt{x}}$.

14. $\sqrt{(2+a)^2+(2-a)x}+\sqrt{(2-a)^2+(2+a)x}=2a$.

15. $\dfrac{1}{x}+\dfrac{1}{2}=\sqrt{\dfrac{1}{4}+\dfrac{3}{2x}\sqrt{\dfrac{8}{9}+\dfrac{4}{9x^2}}}$.

16. $\dfrac{\sqrt{2-\sqrt{x}}}{\sqrt[4]{x+12}}=\dfrac{\sqrt[4]{x-4}}{\sqrt{2+\sqrt{x}}}$.

17. $\sqrt[3]{19+x^2-4x}=\dfrac{5-x}{\sqrt[3]{5-x}}$.

255. Since quantities with fractional exponents can be written in the radical form and *vice versa*, the student is at liberty, in any particular example, to employ the most convenient of the two forms. This should be borne in mind in connection with the examples which follow. These examples are selected from the algebraic reductions occurring in calculus.

PROMISCUOUS EXAMPLES IN THE USE OF FRACTIONAL EXPONENTS AND RADICALS.

Simplify the following:

1. $m\sqrt{1+\left(\frac{n}{m}\right)^{\frac{1}{2}}}-n\sqrt{1+\left(\frac{m}{n}\right)^{\frac{1}{2}}}$.

2. $(-1-\sqrt{5}-\sqrt{-10+2\sqrt{5}})^5$.

3. $\left(\frac{x^{b+c}}{x^{c-a}}\right)^{\frac{1}{a-b}}\left(\frac{x^{c+a}}{x^{a-b}}\right)^{\frac{1}{b-c}}\left(\frac{x^{a+b}}{x^{b-c}}\right)^{\frac{1}{c-a}}$.

4. $\dfrac{\left[x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}}\right)^2\div\frac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}\right]^2}{\left(x^{\frac{1}{2}}y^{-\frac{1}{4}}\sqrt{x^{\frac{1}{3}}y^{-\frac{1}{2}}\sqrt{y^{\frac{1}{2}}}}\right)^3}$.

5. $\dfrac{1+(y+\sqrt{y^2-1})^2}{1+y(y^2-1)^{-\frac{1}{2}}}$.

6. $\dfrac{e^y(e^y+2)^{-\frac{1}{2}}+e^y(e^y-2)^{-\frac{1}{2}}}{\sqrt{e^y+2}+\sqrt{e^y-2}}$.

7. $$m\left[\frac{\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}(x+m)^{-\frac{1}{2}}}{x^{\frac{1}{2}}+(x+m)^{\frac{1}{2}}}\right]+\tfrac{1}{2}(mx+x^2)^{-\frac{1}{2}}(m+2x).$$

8. $$\frac{x+\sqrt{1-x^2}-x\left\{1-x(1-x^2)^{-\frac{1}{2}}\right\}}{(x+\sqrt{1-x^2})^2}.$$

9. $$\frac{9x^3(x^3+1)^{\frac{2}{3}}-(3x^3+2)\left[(x^3+1)^{\frac{2}{3}}+2x^3(x^3+1)^{-\frac{1}{3}}\right]}{x^2(x^3+1)^{\frac{4}{3}}}.$$

10. $$\frac{[\sqrt{x^2+a^2}-\sqrt{x^2+b^2}]\,[(x^2+a^2)^{-\frac{1}{2}}+(x^2+b^2)^{-\frac{1}{2}}]}{[\sqrt{x^2+a^2}-\sqrt{x^2+b^2}]\,[\sqrt{x^2+a^2}+\sqrt{x^2+b^2}]}$$
$$-\frac{[\sqrt{x^2+a^2}+\sqrt{x^2+b^2}]\,[(x^2+a^2)^{-\frac{1}{2}}-(x^2+b^2)^{-\frac{1}{2}}]}{[\sqrt{x^2+a^2}-\sqrt{x^2+b^2}]\,[\sqrt{x^2+a^2}+\sqrt{x^2+b^2}]}.$$

11. $$\frac{\dfrac{(1+\sqrt{x})^{\frac{1}{2}}\left(-\dfrac{1}{2\sqrt{x}}\right)}{2(1-\sqrt{x})^{\frac{1}{2}}}-\dfrac{(1-\sqrt{x})^{\frac{1}{2}}\left(\dfrac{1}{2\sqrt{x}}\right)}{2(1+\sqrt{x})^{\frac{1}{2}}}}{\left(1+x^{\frac{1}{2}}\right)^2-\left(1-x^{\frac{1}{2}}\right)^2}.$$

12.
$$\frac{\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}\left\{\dfrac{1}{2(1+x)^{\frac{1}{2}}}+\dfrac{1}{2(1-x)^{\frac{1}{2}}}\right\}}{\left[\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}^2-2\right]^{-1}}$$
$$+\frac{\left\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}\right\}\left\{\dfrac{1}{2(1+x)^{\frac{1}{2}}}-\dfrac{1}{2(1-x)^{\frac{1}{2}}}\right\}}{\left[\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}^2-2\right]^{-1}}.$$

Prove the following:

13. $$n\left(\frac{x}{1+\sqrt{1-x^2}}\right)^{n-1}\left(\frac{1+\sqrt{1-x^2}+x^2(1-x^2)^{-\frac{1}{2}}}{(1+\sqrt{1-x^2})^2}\right)$$
$$=\frac{ny}{x\sqrt{1-x^2}},\text{ when } y=\left(\frac{x}{1+\sqrt{1-x^2}}\right)^n.$$

14. $$n(n-1)(e^x+e^{-x})^{n-2}(e^x-e^{-x})^2+n(e^x+e^{-x})^n$$
$$=n^2y-4n(n-1)y^{\frac{n-2}{n}},\text{ when } y=(e^x+e^{-x})^n.$$

Suggestion. $(e^x-e^{-x})^2=(e^x+e^{-x})^2-4.$

CHAPTER XVII.

QUADRATIC EQUATIONS.

INVOLVING ONE UNKNOWN QUANTITY.

256. **A quadratic equation** is an equation of the second degree.

257. **A pure quadratic equation** is a quadratic equation containing the second power only of the unknown quantity; as $2x^2 = 9$.

258. **An affected quadratic equation** is a quadratic equation containing both the first and second powers of the unknown quantity; as $3x^2 - 5x = 7$.

PURE QUADRATIC EQUATIONS.

259. Pure quadratic equations differ from simple equations in that the square instead of the first power of the unknown quantity is used.

We have found that a simple equation can be reduced to the form

$$x = a.$$

By exactly similar steps, a pure quadratic equation can be reduced to the form

$$x^2 = a.$$

The values of x can then be found by extracting the square root of each member of the equation. If a is a negative quantity, the values of x will be imaginary; if a is positive, there will evidently be two real values of x, equal numerically but of opposite sign. If the law of signs were followed strictly, the double sign would appear before each member of the equation, resulting from the extraction of the root; since however, no more values of x will be found, by using the double sign before each member, than if it were placed before the second member only, this latter plan is adopted.

The values of the unknown quantity obtained by the solution, are called the *roots* of the equation. This use of the word root must be carefully distinguished from its use in evolution.

Illustrative Example.

Solve $\frac{8}{1-2x}+\frac{8}{1+2x}=25.$ (1)

Clearing of fractions, $8+16x+8-16x=25-100x^2.$ (2)

Collecting terms, $100x^2=9.$ (3)

Dividing by 100, $x^2=\frac{9}{100}.$ (4)

Extracting square root, $x=\pm\frac{3}{10}.$ (5)

EXAMPLES.

Solve the following:

1. $\frac{7}{x-3}-\frac{7}{x+3}=6.$

2. $2x(x-3)=5(8-\frac{6}{5}x).$

3. $\left(8x - \frac{1}{4}\right)^2 = 16\frac{1}{16} - 4x.$

4. $(x-4)(x-2) + 2(x-7)(x-8) =$
$6(x-3)^2 + 18.$

5. $a\left(ax - \frac{8}{ax}\right) = \frac{8a - 7a^2x}{2} - 4a.$

6. $17 - \sqrt{3x^2 + 6} = 8.$

7. $x + \sqrt{x^2 + b^2} = \frac{2b^2}{\sqrt{x^2 + b^2}}.$

8. $\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}} = 2.$

9. $x\sqrt{1+9x^2} = 4 - 3x^2.$

10. $\frac{1}{a - \sqrt{b - x^2}} - \frac{1}{a + \sqrt{b - x^2}} = \frac{\sqrt{3b}}{a^2 - b + x^2}.$

11. $\sqrt{2+x+x^2} + \sqrt{2-x+x^2} = 2\sqrt{2} + 2.$

12. $\frac{3}{(x-1)(x+2)} + \frac{1}{(x-1)(3-x)} = \frac{2}{x-3}.$

13. $\frac{6x^2+32}{x^2+4} - \frac{2x^2+7}{x^2+1} = 4.$

14. $\frac{x+\sqrt{x^2-8}}{x-\sqrt{x^2-8}} = 2.$

15. $\frac{1}{x}\sqrt{m^2-x^2} - \frac{x}{m} + \frac{m}{x} = 0.$

16. $\frac{1}{2x^2-5x-3} - \frac{1}{2x^2+7x+3} = \frac{a}{2bx+b} + \frac{2bx-b}{ax^2-9a}.$

17. $4x(x-bx)+4x-a^2=4x(1-cx-bx)$.

18. $\sqrt{\frac{x}{2}-4}-\sqrt{\frac{x}{2}+4}=\sqrt{x-6}-\sqrt{x+6}$.

AFFECTED QUADRATIC EQUATIONS.

SOLUTION BY COMPLETING THE SQUARE.

260. Every affected quadratic equation can be reduced to the form

$$ax^2+bx+c=0,$$

in which x represents the unknown quantity and a, b and c the constants occupying these respective positions in the reduced form. Translated into common language this form reads: a constant into the square of the unknown quantity, plus a second constant into the unknown quantity, plus a third constant, equals zero. These constants, in particular examples, may be literal or numerical, integral or fractional, polynomial or monomial quantities.

The student should be able to readily reduce any quadratic equation to this standard form. To this end the following examples are given:—

Illustrative Example.

Reduce $ax^2+3x+7+bx^2-5x-2=8$ to the form $ax^2+bx+c=0$.

$$\begin{aligned}(a+b)x^2-2x-3=0, \text{ where } a&=a+b\\ b&=-2\\ c&=-3.\end{aligned}$$

EXAMPLES.

Reduce to the form $ax^2+bx+c=0$, the following:

1. $x^2-ax-x=-a$. 2. $x+\frac{1}{m}=m+\frac{1}{x}$.

3. $x^2-x\sqrt{2}=ax-bx^2+5$.

4. $\frac{x}{\sqrt{x+1}}+\frac{\sqrt{x+1}}{x}=(a-b)\sqrt{x+1}$.

5. $\frac{x+1}{x+2}=\frac{x+2}{x-5}+\frac{1}{10}$.

6. $\frac{1}{x+\sqrt{3-x^2}}+\frac{1}{x-\sqrt{3-x^2}}=m-n$.

7. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=\frac{10}{3}$.

8. $(x+1)\sqrt{3}=(x-1)(x-a)$.

261. After a quadratic equation has been reduced to the form $ax^2+bx+c=0$ the two members of the equation can be multiplied by such a quantity as will make the first term of the equation a perfect square. The third or absolute term can be transposed to the second member. If the multiplier employed is a the equation will then become $a^2x^2+abx=-ac$. The first member of the equation may now be regarded as the first two terms of the square of a binomial, which square, when completed, will be composed, as is the square of any binomial, of the square of the first term of the binomial, plus twice the algebraic product of the first and second terms, plus the square of the second term. It follows that the square root of the first term of the equation will give the first term of the binomial root, and that twice this first term of the root divided into the second term of the equation will give the second term of the root. The square of this second term of the root is the third or missing term of the power, and added to each member of the equation will render the first member a perfect square. This process is called *completing the square.*

The square root of each member of the equation thus formed can be extracted and the resulting equation readily solved for the values of the unknown quantities.

The steps taken are embodied in the following

RULE.

1. *Reduce the equation to the form* $ax^2 + bx + c = 0$.

2. *Multiply this equation by such quantity as will render its first term positive and a perfect square; transfer the third or absolute term to the second member of the equation.*

3. *Add to each member of the equation the square of the quotient obtained by dividing the second term of the first member by twice the square root of the first term.*

4. *Extract the square root of each member of the equation, placing the double sign* $\pm$ *before the second member.*

5. *Solve the resulting equation.*

Remarks on Examples.

1. The first term of an equation of the form $ax^2 + bx + c = 0$ can be made a perfect square by multiplying the equation by any one of the following quantities: $\frac{1}{a}$, a, $4a$, $9a$, $16a$, etc. It follows that in put-

ting any equation into proper form for completing the square, the following are some of the multipliers which can be used:

(a) The reciprocal of the coefficient of x^2.
(b) The coefficient of x^2.
(c) Four times the coefficient of x^2.
(d) Nine times the coefficient of x^2.
(e) Sixteen times the coefficient of x^2.

The multipliers most commonly used are (a) (b) and (c).

2. If the second member of the equation resulting from extracting the square root, is a polynomial, care should be taken to inclose it in a parenthesis before prefixing the double sign.

3. If the work is complicated or there is danger of mistake because of inexperience, the student is advised to decompose the double sign and solve the two resulting equations separately.

4. If the double sign is retained, it will, in general, be found most convenient, in transposing the second or absolute term of the binomial to the second member, to place it *before* the term which has the double sign.

5. The student should form the habit of dealing with the parts of the double sign in a particular order. This will prove a great convenience to him in reviewing his work. It is customary to give attention to the plus sign first.

Illustrative Examples.

1. Solve $7x = 6x^2 + 2$.

Reducing to the standard form $\qquad 6x^2 - 7x + 2 = 0 \qquad (1)$

Multiplying by 6, $36x^2 - 42x + 12 = 0.$ (2)

Transposing 12, $36x^2 - 42x = -12.$ (3)

Dividing $-42x$ by 2 $(6x)$ and adding its square to both members, $36x^2 - 42x + \left(\frac{7}{2}\right)^2 = \frac{1}{4}.$ (4)

Extracting square root of each member,

$$6x - \frac{7}{2} = \pm\frac{1}{2}. \quad (5)$$

$$6x = \frac{7}{2} \pm \frac{1}{2}. \quad (6)$$

$$6x = 4 \text{ or } 3. \quad (7)$$

$$x = \frac{2}{3} \text{ or } \frac{1}{2}. \quad (8)$$

2. Solve $(x-3)(x+4) = (x-7)\sqrt{3}.$

Expanding, $x^2 + x - 12 = x\sqrt{3} - 7\sqrt{3}.$ (1)

Reducing to standard form,

$$x^2 + (1-\sqrt{3})x - 12 + 7\sqrt{3} = 0. \quad (2)$$

Transposing the absolute terms,

$$x^2 + (1-\sqrt{3})x = 12 - 7\sqrt{3}. \quad (3)$$

Completing the square,

$$x^2 + (1-\sqrt{3})x + \left(\frac{1-\sqrt{3}}{2}\right)^2 = \frac{52 - 30\sqrt{3}}{4}. \quad (4)$$

Extracting the square root,

$$x + \frac{1-\sqrt{3}}{2} = \pm\frac{3\sqrt{3}-5}{2}. \quad (5)$$

$$x = \frac{\sqrt{3}-1}{2} \pm \frac{3\sqrt{3}-5}{2}. \quad (6)$$

$$x = 2\sqrt{2} - 3 \text{ or } 2 - \sqrt{3}. \quad (7)$$

EXAMPLES.

Solve the following:

1. $4x^2+6x+\frac{5}{4}=0.$ 2. $3x^2+4x=7.$

3. $7x^2-x-1=0.$ 4. $4x-6x^2=-\frac{7}{2}.$

5. $32x^2-16x-6=0.$ 6. $27x^2=5+6x.$

7. $ax^2+bx+c=0.$ 8. $5mx^2-\frac{9c^2}{m}=-4cx.$

9. $x-3dx^2=\frac{1}{16d}.$ 10. $18ax^2-7x+\frac{1}{3a}=0.$

11. $(x+1)(x-4)=(3-2x)\sqrt{2}-8.$

12. $x^2-x\sqrt{5}=3x+8.$

SOLUTION BY FORMULA.

262. Every affected quadratic equation can be reduced, as already stated, to the form $ax^2+bx+c=0$. The solution of this equation gives us the general answer or *formula*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Since this is the answer to the general example, the answer to any particular example may be obtained by substituting in this general answer or *formula*, the values of a, b, and c, in the particular example.

Suggestion. It is desirable that the student memorize the formula. He should also bear in mind that it is derived by solving the general example, under the method of completing the square. Remembering this, he can readily derive the formula if, at any time, he should forget it. The essential thing in the solution of affected quadratics is a thorough understanding of the principles upon which the *completion of the square* is based. If these are remembered, any ingenious student can work out the detail steps for the solution of either the general or a particular example.

Remarks on Examples.

1. In reducing a particular example to the form $ax^2+bx+c=0$, it is convenient, but not necessary, to make the sign of the first term positive.

2. The student should not fail to take account of the signs of the known quantities, in substituting in the formula.

Thus,

Solve $2x^2-3x-2=0$, in which $a=2$, $b=-3$, $c=-2$.

$$\text{Then, } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-3)\pm\sqrt{(-3)^2-(4.2.-2)}}{2.2}$$

$$=\frac{3\pm\sqrt{25}}{4}=\frac{3\pm5}{4}=2\text{ or }-\frac{1}{2}.$$

Illustrative Example.

Solve $mx^2+px=cx-1$.

Reducing to the standard form,

$$mx^2+(p-c)x+1=0.$$

Then, $a = m, b = p - c, c = 1$.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{c - p \pm \sqrt{(p-c)^2 - 4m}}{2m}.$$

EXAMPLES.

Solve the following equations:

1. $6x^2 + x = 2$. 2. $35 + 12x^2 = 43x$.

3. $2x - 5x^2 = 7$. 4. $3x - 2x^2 = 0$.

5. $361x^2 + 38x = 399$. 6. $7a^2x^2 - 9ax = 10$.

7. $(a^2 - b^2)x^2 + 2bx = 1$.

8. $6abx^2 + (9b^2 - 4a^2)x = 6ab$.

9. $5bx^2 - bcx + cd = 5dx$.

10. $5m - cnx^2 + mcx = 5nx$.

DISCUSSION OF THE FORMULA.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

263. The discussion of a formula consists, as already shown, in substituting particular values for its known, or arbitrary, quantitative symbols, and then interpreting the result. In this article we shall conconsider the cases where such values are assigned to a, b and c as will make:

1. $b^2 - 4ac > 0$.
2. $b^2 - 4ac = 0$.
3. $b^2 - 4ac < 0$.

Case I. $b^2-4ac>0$.

When $b^2-4ac>0$, the quantity under the radical is positive, and hence its square root is a real quantity. This is added to the first term of the numerator, in determining one root, and substracted from it, in determining the other. The two roots are, therefore, *real and unequal.*

Case II. $b^2-4ac=0$.

Since the quantity under the radical equals zero, the formula, in this case, reduces to $x=\frac{-b\pm 0}{2a}=\frac{-b}{2a}$. The two roots are evidently *real and equal.*

Case III. $b^2-4ac<0$.

In this case the quantity under the radical is negative, its square root is imaginary, and therefore the roots determined by the formula are *imaginary roots.*

264. It is always possible, and often convenient, in reducing an affected quadratic equation to the form $ax^2+bx+c=0$ to make the coefficient of x^2 equal one. The equation can then be written

$$x^2+bx+c=0.$$

The formula of the previous article reduces, under this condition, (a being equal to one) to

$$x=\frac{-b\pm\sqrt{b^2-4c}}{2}.$$

Separating the double sign, for convenience, and representing the two roots respectively by x_1 and x_2 we have

$$x_1=\frac{-b+\sqrt{b^2-4c}}{2}.$$

$$x_2=\frac{-b-\sqrt{b^2-4c}}{2}.$$

It is proposed to discuss these two roots for all possible values of b and c, other than infinity. The discussion will be complete if we consider fully each of the three following cases:—

1. *When b has any negative value,*
2. *When b equals zero.*
3. *When b has any positive value,*

discussing under each case the three sub-cases, arising from assigning to c the same classes of values which are assigned to b.

Case I. *When b is negative.*

Sub-case I. *When c also is negative.*

Under these conditions it follows:

(a) The first term in the numerator of each root is positive.

(b) Each term of the quantity under the radical is positive, and the quantity takes the form b^2+4c.

(c) Since $b^2+4c > b^2$, $\sqrt{b^2+4c} > b$ and therefore the sign of the numerator of each root, and consequently the sign of each root, is determined by the sign of the radical.

It follows that both roots are real; the first is positive, the second negative, and the positive root is numerically the greater.

Sub-case 2. *When c equals zero.*

When $c=0$, (b remaining negative)

$$x_1 = \frac{b+b}{2} = \frac{2b}{2} = b$$

$$x_2 = \frac{b-b}{2} = \frac{0}{2} = 0.$$

Sub-case 3. *When c is positive.*

When $c > 0$, (b remaining negative), we have:

(a) The first term in each numerator is positive.

(b) The first term of the quantity under the radical is positive, the second term negative and the quantity retains the form $b^2 - 4c$.

(c) When $4c < b^2$, $b^2 - 4c$ is positive and $\sqrt{b^2 - 4c}$ is real, but $b^2 > b^2 - 4c$ therefore $b > \sqrt{b^2 - 4c}$ and hence the sign of each root will be determined by the sign of the first term in its numerator, which term we have found to be positive. The roots then in this sub-case, when $4c < b^2$, are real and positive and the first root is numerically the greater.

(d) When $4c = b^2$, $b^2 - 4c = 0$ and the roots reduce to

$$x_1 = +\frac{b}{2}$$

$$x_2 = +\frac{b}{2}.$$

The roots are real, negative and equal.

(e) When $4c > b^2$, $b^2 - 4c$ is negative, $\sqrt{b^2 - 4c}$ is imaginary and the two roots are consequently imaginary.

Case II. *When b equals zero.*

Sub-case 1. *When c is negative.*

When $c < 0$ (and $b = 0$),

$$x_1 = +\sqrt{c}.$$

$$x_2 = -\sqrt{c}.$$

The two roots are real, and equal numerically, but with opposite signs.

Sub-case 2. *When c equals zero.*

When $c = 0$ (and $b = 0$).
Each root evidently equals zero.

Sub-case 3. *When c is positive.*

When $c > 0$ (and $b = 0$),

$$x_1 = +\sqrt{-c}.$$
$$x_2 = -\sqrt{-c}.$$

The two roots are imaginary.

Case III. *When b is positive.*

The following are evidently the sub-cases:

Sub-case 1. *When c is negative.*

Sub-case 2. *When c equals zero.*

Sub-case 3. *When c is positive.*

The student is expected to discuss fully each of these sub-cases. This he can do if he has come to understand the real significance of the discussion of an algebraic formula, and has mastered the details of the discussion of cases I and II.

265. Remarks on Examples.

1. As it is exceedingly important in higher mathematical work that the student shall be able to apply the results of a general discussion to particular examples, he is urged to exercise special care in handling the practice work of this kind here given.

2. When the discussion of a particular example consists of independent steps, the student, in class presentation, should state and explain each step separately, in the order given in the example.

EXAMPLES.

Without solving, determine whether the roots of each of the following equations are real and unequal, real and equal, or imaginary:

1. $7x^2 - 14x + 7 = 0.$

2. $5x^2 - x + 8 = 0.$

3. $2x^2 = 5x + 6.$

4. $x + 5 + \dfrac{2}{x+1} = 1.$

Without solving, determine what value of the arbitrary constant will make the roots of each of the following equations, real and unequal, real and equal, or imaginary:

5. $2x^2 - mx + 18 = 0.$

6. $3x^2 + 6x + c = 0.$

7. $9x^2 + 2gx - 1 = 0.$

8. Find b in terms of m such that the equation $3x^2 + 2bx + 5m = 0$ shall have real and unequal roots.

9. Prove that if $c = \frac{1}{2}\sqrt{a}$, the equation $2ax^2 + 8cx + 2 = 0$ will have equal roots.

10. Prove that the condition that the roots of $3m^2x^2 - 2nx + 12 = 0$ shall be imaginary, is $n < 6m$ numerically.

11. Find the condition that the equation $5x^2 - 12x + 9 = r^2$ shall have equal roots.

Without solving, determine the following regarding the roots of each equation:

(*a*) Whether real or imaginary.
(*b*) Sign of each root.
(*c*) Relative numerical value of the two roots.

12. $4x^2 - 5x + 1 = 0$.

13. $5x^2 - x + 6 = 0$.

14. $9x^2 + 6x + 1 = 0$.

15. $4x^2 - 3x = 0$.

16. $7x^2 - 6x = 4$.

17. $6x^2 + 8x = 1$.

18. $5x^2 + 2x = 0$.

19. $4x^2 - 12x + 9 = 0$.

20. $2x^2 + 6x + 3 = 0$.

MISCELLANEOUS EXAMPLES.

266. In solving the following examples, the student should use his judgment as to whether he should *complete the square* or use the *formula*.

Solve the following equations:

1. $\frac{9}{x} + \frac{x}{3} = 4$.

2. $2x - \frac{6 - 3x}{2x} = 2\frac{1}{2}$.

3. $x - \frac{x - 3}{x - 4} = 2x - \frac{5x - 2}{3}$.

4. $\sqrt{(x+4)(1-x)} = 2x + 3$.

5. $\frac{(x+3)^2}{4} + \frac{x+3}{2} - \frac{3}{4} = 0$.

6. $x + \sqrt{2x - 5} = 4.$

7. $\dfrac{1}{m+n+x} = \dfrac{m}{x} - \dfrac{n}{x} - 1.$

8. $\dfrac{1}{x+3} + \dfrac{1}{x-3} + \dfrac{1}{9-x^2} = 1.$

9. $3\sqrt{x} + \dfrac{3}{\sqrt{x}} = 2\sqrt{-3}.$

10. $\sqrt{x-3} - \sqrt{2x-8} + 1 = 0.$

11. $\dfrac{x^2-3x+5}{x^2+5x+6} = \dfrac{x^2-7x-6}{x^2-x-6} - \dfrac{7}{x+3}.$

12. $\dfrac{3x^2 - 11x + 4}{x-3} - \dfrac{2x^2 + 6x + 4}{x+4} = x - 3\frac{1}{2}.$

13. $\sqrt{\dfrac{x-3}{x+5}} + \sqrt{\dfrac{x-5}{x+3}} = \sqrt{\dfrac{x-2}{x-1}} + \sqrt{\dfrac{x-1}{x-2}}.$

14. $2x - 6 - (x-3)\}x - 2 - 4(x-5)\{ = 0.$

15. $4(x-1) + \sqrt{2x + \dfrac{3}{2}} = 3.$

16. $\sqrt{(x-1)(x+2)} + 4\sqrt{\dfrac{x-1}{x+2}} = \sqrt{(x+2)(x+6)}.$

17. $\dfrac{b}{(x-a)\ (b-x)} - \dfrac{a}{(x-c)\ (a-x)} = \dfrac{x}{(x-b)\ (c-x)}.$

18. $(7 + 4\sqrt{3})x^2 + (2 + \sqrt{3})x = \dfrac{9}{2} - 2\sqrt{3}.$

19. $\dfrac{x + b + \sqrt{x^2 - b^2}}{x + b - \sqrt{x^2 - b^2}} = \dfrac{x + b}{2b}.$

20. $\frac{x^2-1}{x} = \frac{1-m}{n} - \frac{n}{1-m}.$

21. $(x-2)(x+3) = (4-x)\sqrt{2}.$

22. $x^2 - ax = (a-x)\sqrt{b}.$

23. $\frac{x^2}{ab} - \left(1 + \frac{1}{ab}\right)x = \frac{a-1}{b} - a.$

24. $\frac{3m+x}{ax+3mx} - \frac{a-x}{a^2-9m^2} + \frac{x-3m}{3mx-ax} = 0.$

25. $x^2 - \frac{am}{b^2} + \frac{1}{4} = \frac{b^2-m^2}{4b^2} - \frac{(2a-m)x}{b}.$

26. $\sqrt{x-1} + \sqrt[4]{x-1} = 2.$

27. $$\frac{1}{(x+3)(x-2)} + \frac{1}{(x+3)(x+2)} = \frac{a}{b(x+3)} - \frac{b(x-3)}{a(x+2)(x-2)}.$$

28. $\frac{x^2-x-12}{x^2+8x+15} - \frac{x^2-11x-26}{x^2+7x+10} = x-3.$

29. $(a^2-4ab+3b^2)x^2 + 4b^2x = a^2 - b^2.$

30. $\sqrt{x+m} = 2a^2 - a\sqrt[4]{x+m}.$

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

267. 1. Find two consecutive numbers whose product is 20.

2. Find two numbers whose difference is 5, such that the sum of their squares exceeds their product by 75.

3. The length of a field exceeds its breadth by 60 rods, and the area of the field is 10 acres. What are the dimensions of the field?

4. The denominator of a certain fraction exceeds the numerator by 1, and if the fraction were inverted, it would equal the original fraction plus $\frac{5}{6}$. What is the fraction?

5. A man bought a set of books for $48. If there had been 8 less books in the set, they would have cost him $1 more per volume. How many books were there in the set?

6. A crew rows 24 miles up stream and back again in 5 hours. The rate of the crew in still water is 2 miles per hour more than 4 times that of the current. What is the rate of the current and of the crew in still water?

7. A rectangular room is paved with 1,350 square tiles. If the tiles were 2 inches wider and 4 inches longer, it would require 720. What is the size of each tile?

8. At a certain banquet, the bill of the party in attendance was $30. As five of the party were guests of the others, each of the entertainers paid 50 cents more than he otherwise would have done. What was the price per plate of the banquet?

9. A man bought a number of horses for $1,800 and then by selling them at $175 apiece, gained the cost of two horses. How many did he buy?

10. A bookseller bought a number of books for $120, which he sold at an advance of 50 cents per

volume. He sold all but 4 and found he had $20 more than all the books cost him. How many did he buy?

11. A man bought a number of sheep for $80. If each had cost one dollar less, he would have had 5 times as many sheep as each would have cost him. How many sheep did he buy?

12. A number consisting of two digits, has its tens digit 4 less than its units digit. The square of the number exceeds the number with its digits interchanged by 174. What is the number?

13. A train was scheduled to make a journey of 1,200 miles in a certain number of hours, but after traveling 12 hours at its usual rate, an accident delayed the train 45 minutes, after which it proceeded at a rate of 30 miles per hour and in consequence was $6\frac{3}{4}$ hours late. What was the scheduled time for the journey?

14. A rectangular park whose length exceeds its breadth by 80 feet, has a walk around it whose width is $\frac{1}{20}$ the width of the portion inclosed, and the area of the walk is 34,000 sq. ft. What is the width of the walk?

15. A man sold a cow at an advance of as many per cent as the cow cost in dollars. With this he bought a number of pigs at $1.50 each, and found that he had 13 less than the number of dollars received for the cow. What did the cow cost him?

16. A and B start to ride on bicycles from P to Q, 150 miles apart. A starts first and rides at a uniform rate. When he has ridden 3 hours, B starts and rid-

ing at a rate of 2 miles an hour faster than A, overtakes him when 30 miles from Q. What was the rate of each in miles per hour?

17. A broker bought a certain number of shares of stock for $1,500, and a few days later sold them at an advance of $10 a share. With the money received, he bought shares of another kind costing him $50 less per share than the first cost him, and found he had 6 more shares than in the first purchase. How many shares did he first buy?

18. A man having an appointment 24 miles in the country, found no carriage at the station to meet him. After walking 2 hours, he met the carriage, which had been injured by an accident to such an extent that it could be driven at but $1\frac{1}{2}$ times the rate at which the man was walking, and at $\frac{2}{3}$ the rate at which it would have traveled had there been no accident. In consequence, he was late $\frac{1}{2}$ as many hours as the number of miles which he walked in an hour. How far was the place where he met the carriage from the station?

19. Two adjoining farms, each in the form of a square, have each a side along the same highway. 560 rods of fence are necessary to inclose both farms in a single lot, and the difference in the areas of the fields is $22\frac{1}{2}$ acres. What is the area of each field?

20. Two men having sold cattle, found they had both received the same sum, although there was a difference of 5,000 pounds in the weights of cattle sold by the two men. The two prices differed by half a cent per pound. If both had sold at the higher price, they would have received $4,750 for all the

cattle. What was the price per pound received by each?

21. A man started to walk from M to N at the rate of 4 miles an hour; after one-half an hour, another started from N to M., driving as many miles an hour as the number of hours it would take the first to walk the distance. They were 6 miles apart when the first had traveled $\frac{1}{4}$ as many hours as the second traveled miles in an hour. How far apart were M and N?

22. A man has an income of $5,000 a year. His taxes are a certain per cent of his income and the per cent of his insurance less by two than the per cent of his taxes. After paying both these, he spent a per cent of the remainder equal to 10 times the per cent of his insurance, leaving $3,760. What was the per cent of his tax?

23. A man has two casks of equal size, one containing wine and the other water, and each $\frac{3}{4}$ full. He took from the cask of water 10 gallons which he put in the wine cask and then put as many more gallons of water in the wine cask as the per cent of water in the wine cask. The wine cask was then $\frac{11}{12}$ full. What does each cask hold when full?

24. The distance from A to B is 940 miles. A train from B starts west an hour after one has started east from A and travels at a rate of 40 miles an hour; the east bound train travels $2\frac{4}{13}$ times as many miles in an hour as the train from B has traveled hours when they meet. How far from B was the place of meeting?

EQUATIONS SOLVED LIKE QUADRATICS.

268. We have found that any affected quadratic equation can be reduced to the form $ax^2 + bx + c = 0$. Translated into ordinary language, this may be read: *Any known, into the square of the unknown quantity; plus any known, into the first power of the unknown quantity; plus any known quantity, equals zero.* In the strictly quadratic equation, or equation of the second degree, the unknown quantity is represented by a single letter or symbol. The unknown quantity may, however, be represented by a monomial or polynomial *expression* instead of by a single letter, and the equation still be *quadratic in form.*

Thus, let x be replaced by $2y^{-2}$, and we will have $a(2y^{-2})^2 + b(2y^{-2}) + c = 0$.

Again, let $x^2 + 3x + 1$ be the unknown expression or quantity appearing in the form, and we have $a(x^2 + 3x + 1)^2 + b(x^2 + 3x + 1) + c = 0$.

It is evident that many equations of a higher degree than the second can be reduced to the quadratic form. The student should learn to make these reductions readily and understandingly. The following practice examples are given in which he is asked to specify, in each case, the unknown expression (or quantity) dealt with in the form.

EXAMPLES.

Reduce the following to the quadratic form:

1. $2x^6 + 1 = 3x^3$.

2. $\sqrt{x^2 + 2x} = 2x^2 + 4x + 9$.

3. $x^{-3} = 4x^{-6} + 7$.

4. $5x^{\frac{3n}{7}} + 6 = 4x^{\frac{6n}{7}}$.

5. $\sqrt[4]{1 - x^2} = 1 + \sqrt{1 - x^2}$.

6. $(x^2 + 3x + 1)^2 + 6x^2 + 18x + 19 = 0$.

269. Any equation in the quadratic form can be solved for the unknown expression involved, either by completing the square or by the use of the formula.

Remarks on Examples.

1. The most important step in the solution of equations of this class is the reduction of the equation to the quadratic form. In choosing the expression to be treated as the unknown quantity, and in deciding upon the proper steps to be taken in the reduction, much must be left to the ingenuity of the student. The following suggestions may, however, be helpful:

(a) If the equation contains a radical expression, it often happens that, by a proper arrangement of the other terms and the addition of the right quantity to each member, an expression can be formed which will be equal to the quantity under the radical or to that quantity multiplied by some known factor, while the remaining terms of the equation will contain known quantities only. The equation can then be readily reduced to the desired form. (See first illustrative example.)

(b) When the equation does not contain a radical, it is possible that steps similar to those mentioned in (a) will lead to an equation containing, in separate terms, the first and second powers of some unknown expression, the other terms involving only known quantities.

(c) Certain equations of the fourth degree can be reduced to quadratic form by transposing all the terms to the first member and then proceeding, in the ordinary way, to extract the square root of this resulting polynomial. If, at any time in the process of extracting the root, a remainder is obtained which consists of some known quantity times the partial root found, together with terms involving known quantities only, the equation can be put in quadratic form as follows:

(Partial root)^2 + multiple of partial root + known terms $= 0$. (See second illustrative example.)

2. It is sometimes convenient to represent the unknown expression involved, by a single letter, as y. The square of the unknown quantity will then, of course, be represented by y^2. While this device can be employed in all examples, it is especially useful when the unknown expression is a fraction and both it and its reciprocal are contained in the equation given for solution. (See third illustrative example.)

3. In the solution of equations reducible to the quadratic form two values will, in general, be obtained for the expression which is treated as the unknown quantity. Placing this expression equal to each of these values, in succession, we have two examples, the solution of which may involve again the use of the quadratic form. It quite often happens that the new examples are of the class $x^{\frac{m}{n}} = a$. In their solution, it is generally best to first extract the mth root of each member and then raise each to the nth power. (See fourth illustrative example.)

Illustrative Examples.

1. $2x^2 + \sqrt{x^2+5x+2} = 32 - 10x$; find x.

Transposing and arranging terms,

$$2x^2+10x+4+\sqrt{x^2+5x+2} - 36 = 0. \quad (1)$$

Reducing to the quadratic form,

$$2(x^2+5x+2)+\sqrt{x^2+5x+2}-36=0. \quad (2)$$

By the formula,

$$\sqrt{x^2+5x+2}=\frac{-1\pm\sqrt{1+288}}{4} \quad (3)$$

$$=\frac{-1\pm 17}{4}=4 \text{ or } -\frac{9}{2}.$$

Squaring, $$x^2+5x+2=16 \text{ or } \frac{81}{4}. \quad (4)$$

From (4), $$\begin{cases} x^2+5x-14=0, \\ x^2+5x-\dfrac{73}{4}=0. \end{cases} \quad (5)$$

By the formula,

$$x=\frac{-5\pm\sqrt{25+56}}{2} \text{ or } \frac{-5\pm\sqrt{25+73}}{2} \quad (6)$$

$$=2,-7 \text{ or } \frac{-5\pm 7\sqrt{2}}{2}.$$

2. $x^4-6x^3+12x^2-9x=28$; find x.

Transposing and proceeding to find the square root,

$$\begin{array}{r|l}
 & x^4-6x^3+12x^2-9x-28 \;\lfloor\underline{x^2-3x} \\
 & x^4 \\ \hline
2x^2-3x & -6x^3+12x^2-9x-28 \\
 & \underline{-6x^3+9x^2} \\
 & 3x^2-9x-28 \\
 & =3(x^2-3x)-28.
\end{array}$$

The original equation can therefore be put in the form, $$(x^2-3x)^2+3(x^2-3x)-28=0. \quad (1)$$

Completing the square,

$$(x^2-3x)^2+3(x^2-3x)+\frac{9}{4}=\frac{121}{4}. \quad (2)$$

Extracting the square root, $x^2 - 3x + \frac{3}{2} = \pm \frac{11}{2}$. (3)

From (3), $x^2 - 3x = 4 \text{ or} - 7$. (4)

Completing the square, $x^2 - 3x + \frac{9}{4} = \frac{25}{4} \text{ or} - \frac{19}{4}$. (5)

Extracting the square root, $x - \frac{3}{2} = \pm \frac{5}{2} \text{ or } \pm \frac{1}{2}\sqrt{-19}$. (6)

From (6), $x = 4, -1, \text{ or } \frac{3 \pm \sqrt{-19}}{2}$. (7)

3. $\frac{4x}{x^2+1} + \frac{x^2+1}{x} = 4$; find x.

Substituting y for $\frac{x}{x^2+1}$, $4y + \frac{1}{y} = 4$. (1)

From (1), $4y^2 - 4y + 1 = 0$. (2)

By the formula, $y = \frac{4 \pm \sqrt{16 - 16}}{8} = \frac{1}{2}$. (3)

Replacing y by $\frac{x}{x^2+1}$, $\frac{x}{x^2+1} = \frac{1}{2}$. (4)

From (4), $x^2 - 2x + 1 = 0$. (5)

By the formula, $x = 1$. (6)

4. $(x^2 - x + 1)^{-\frac{6}{5}} = 4(x^2 - x + 1)^{-\frac{3}{5}} - 3$; find x.

Transposing and completing the square,

$(x^2 - x + 1)^{-\frac{6}{5}} - 4(x^2 - x + 1)^{-\frac{3}{5}} + 4 = 1$. (1)

Extracting the square root, $(x^2 - x + 1)^{-\frac{3}{5}} - 2 = \pm 1$. (2)

From (2), $(x^2 - x + 1)^{-\frac{3}{5}} = 3 \text{ or } 1$. (3)

Taking the reciprocal of each member,

$(x^2 - x + 1)^{\frac{3}{5}} = \frac{1}{3} \text{ or } 1$. (4)

Taking the cube root, $(x^2-x+1)^{\frac{1}{5}}=\frac{1}{3}\sqrt[3]{9}$ or 1. (5)

Raising to fifth power, $x^2-x+1=\frac{1}{9}\sqrt[3]{3}$ or 1. (6)

Transposing and completing the square,

$$x^2-x+\frac{1}{4}=-\frac{3}{4}+\frac{1}{9}\sqrt[3]{3} \text{ or } \frac{1}{4}. \quad (7)$$

Extracting the square root,

$$x-\frac{1}{2}=\pm\frac{1}{6}\sqrt{-27+4\sqrt[3]{3}} \text{ or } \pm\frac{1}{2}. \quad (8)$$

$$x=\frac{1}{2}\pm\frac{1}{6}\sqrt{-27+4\sqrt[3]{3}},\ 1, \text{ or } 0. \quad (9)$$

EXAMPLES.

Solve the following:

1. $x^4-14x^2+45=0$.
2. $x^{10}-3x^5+2=0$.
3. $5x^{-2}-6x^{-1}=8$.
4. $8x^{2m}-3x^m=5$.
5. $2x^{-4}-5x^{-2}=12$.
6. $3x^{\frac{1}{3}}-7x^{\frac{1}{6}}=6$.
7. $(x^2-4x)^2-3(x^2-4x)=10$.
8. $x^4+2x^3-13x^2-14x+24=0$.
9. $x+5=4\sqrt{x+5}-3$.
10. $\left(x+\frac{2}{x}\right)^2+3\left(x+\frac{2}{x}\right)=18$.
11. $\frac{x+1}{x-1}+5\left(\frac{x-1}{x+1}\right)=4\frac{2}{3}$.

12. $(x^2+2x-11)^{\frac{3}{2}}-(x^2+2x-11)^3+56=0.$

13. $\sqrt[3]{3x}-\sqrt[3]{9x^2}+2=0.$

14. $2x^2-x+\sqrt{2x^2-x+3}=9.$

15. $(x^2+3)^2=2x^4+17.$

16. $\sqrt{8x}+\sqrt[4]{8x}=6.$

17. $\sqrt{x^2-6x}-5\sqrt[4]{x^2-6x}+6=0.$

18. $x^2+8x+23+28x^{-1}+12x^{-2}=0.$

19. $2x^5+\frac{3}{x^5}+5=0.$

20. $2(x-3)^{-\frac{2}{3}}+3(x-3)^{-\frac{1}{3}}=5.$

21. $3x^2-5x+8\sqrt{3x^2-2x+3}=17-3x.$

22. $5x^{\frac{1}{n}}-4\sqrt[2n]{x}=33.$

23. $\sqrt{6x^2+2x-3}-4x=12x^2-51.$

24. $\frac{x^2-2x}{x+5}-3\left(\frac{x+5}{x^2-2x}\right)+7\frac{5}{8}=0.$

25. $4\sqrt{x}-x^{-\frac{1}{2}}=3.$

26. $4x^4+12x^3+7x^2-3x-2=0.$

27. $4x(2-x)=\sqrt{x^2-2x+10}+1.$

28. $7x^{-\frac{3}{5}}+2x^{-\frac{6}{5}}=22.$

29. $(x+1)^4-2(x+1)^3+3(x+1)^2-2(x+1)=8.$

30. $(x-2)^2-2\sqrt{x}(x-2)=1-2\sqrt{x}.$

31. $6x^{-\frac{2}{7}}-4x^{\frac{2}{7}}=10.$

32. $2(x^2-5x+7)^{-\frac{5}{3}}-(x^2-5x+7)^{-\frac{5}{6}}=1.$

33. $\sqrt{\dfrac{1+x+x^2}{1-x+x^2}} + 3\sqrt{\dfrac{1-x+x^2}{1+x+x^2}} = 4.$

34. $5x^{-\frac{3}{2a}} - 6x^{-\frac{3}{4a}} = 27.$

35. $\sqrt{x^2+3x} + 2\sqrt[4]{x^2+3x} = 2 + 2\sqrt{2}.$

36. $3x^2 + x - 7 + \sqrt{2x^2+3x-5} = \dfrac{34-7x}{2}.$

37. $(x-m^4)^{\frac{3}{2}} + 4m^3(x-m^4)^{\frac{3}{4}} = 5m^6.$

38. $\sqrt{1+\dfrac{12}{x^2-6}} + \sqrt{1-\dfrac{12}{x^2+6}} = \dfrac{5}{2}.$

39. $(3x+1)^2 + 3x^2(3x+1) = 28x^4.$

40. $7x(7-x) = \sqrt{x^2-7x+7} + 41.$

41. $(x-m)^{\frac{1}{2}} + 2\sqrt{n}(x-m)^{\frac{1}{4}} - 3n = 0.$

SOLUTION OF QUADRATIC AND HIGHER EQUATIONS BY FACTORING.

270. Take the equation, $x^2 - 5x + 6 = 0.$

Factoring, $(x-2)(x-3) = 0.$

Any value of x which will reduce either of these factors to zero, will reduce their product to zero, and thus satisfy the equation. Such values are therefore roots of the equation. They can evidently be found by placing each factor in succession equal to zero. In the example given, the roots are 2 and 3.

Any equation which can be reduced to the form

$(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ldots(x-a_n)=0;$

can be solved, in a similar way, by placing the factors in succession equal to zero. The roots obtained will be $a_1, a_2, a_3, a_4, \ldots\ldots a_n$.

271. Proposition. *Every equation containing one unknown quantity can be reduced to the form,*
$(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ldots\ldots(x-a_n)=0,$
in which, as will be shown, a_1, a_2, a_3, ...a_n *are roots of the equation.*

Proof. Every equation of *any degree* involving one unknown quantity, can be reduced to the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}\ldots\ldots p_n=0; \qquad (1)$$

in which p_1, p_2, p_3...p_n are known quantities and n is a positive integer.

It remains to be shown that this equation is reducible to the form given in the proposition.

Let a_1 represent a root* of equation (1). Dividing the first member of this equation by $x-a_1$, we will, directly, obtain a known quantity as a remainder. Denoting this remainder by R and the quotient by Q, equation (1) can be put in the form

$$Q(x-a_1)+R=0. \qquad (2)$$

Since x and a_1, in equation (2) are equal, $R=0$, and the division is exact. It is thus proven that $x-a_1$ is an exact divisor, and consequently a factor of the first member of equation (1), when a_1 is a root of that equation.

It is now evident that we can resolve the first member of equation (1) into two factors one $x-a_1$, and the other a polynomial of a degree one less than equation (1). Values of x which will reduce this second factor to zero will evidently satisfy equation (1) and are therefore roots of that equation. Placing this second factor equal to zero, and dealing with the new equation thus formed as with equation (1), we can resolve the factor into two factors, one $x-a_2$, in which a_2 is a root of the new equation and consequently, as just shown, a root of equation (1), while the other factor will be a new polynomial of a degree one less than the new equation.

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* Cauchy has proven that every equation has a root.

Decomposing this new polynomial factor in a similar way, and noticing that the degree of the factor to be further decomposed, will, at each step, be reduced by one, it is clear that equation (1) will eventually be reduced to the form

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0; \quad (3)$$

in which the first member is composed of n factors of the first degree.

Corollary. *Number of Roots.* Since each of these factors placed equal to zero will give a root of the equation, there will be n roots. There can be no other root, for if x be given a value different from any of the n roots thus found, no factor will reduce to zero and hence the product cannot equal zero.

It follows that *a quadratic equation has two roots and only two.*

Remarks on Examples.

272. 1. In the practice examples given, the factors can, in general, be obtained by the methods set forth in the chapter on factoring. The work in quadratics, however, emphasizes the usefulness of the device of completing the square. Examination of a particular example may reveal that by grouping certain terms, a perfect square will be formed except that the second or third term of such square will be missing. This term can be supplied by the addition and subtraction of the proper quantity. The whole expression can then be thrown into the form of the difference of two squares and thus readily factored. (See third and fourth illustrative examples.)

2. The student should bear in mind that equations must often be changed considerably before they are in shape to be put in factored form. Radicals must be disposed of; terms rearranged, collected or expanded; parentheses removed or introduced and other devices adopted, as the particular case may require. The examples will test the ingenuity, judgment, and courage of the student. Their mastery means algebraic strength and maturity.

3. In solving equations of a high degree it is sometimes advisable not to decompose factors of the second degree, since these can be more easily dealt with by methods already given for solving quadratic equations.

Illustrative Examples.

1. Solve by factoring, $x^4-2x^3-8x^2+9x+18=0$.

Rearranging the terms,

$$x^4-2x^3+x^2-9(x^2-x)+18=0. \quad (1)$$

$$(x^2-x)^2-9(x^2-x)+18=0. \quad (2)$$

Factoring,
$$(x^2-x-6)(x^2-x-3)=0. \quad (3)$$

$$(x-3)(x+2)(x^2-x-3)=0.$$

Placing the factors in succession equal to 0 and solving,

$$x-3=0 \quad \therefore \quad x=3. \quad (4)$$

$$x+2=0 \quad \therefore \quad x=-2. \quad (5)$$

$$x^2-x-3=0 \quad \therefore \quad x=\frac{1\pm\sqrt{1+12}}{2} \quad (6)$$

$$=\frac{1\pm\sqrt{13}}{2}.$$

2. Solve by factoring,

$$\sqrt{x^2-3x+2}-\sqrt{2x^2-10x+14}=\sqrt{x^2-7x+12}.$$

Transposing terms,

$$\sqrt{x^2-3x+2}-\sqrt{x^2-7x+12}=\sqrt{2x^2-10x+14}. \quad (1)$$

Squaring both members,

$$x^2-3x+2-2\sqrt{x^2-3x+2}\sqrt{x^2-7x+12} \quad (2)$$
$$+x^2-7x+12=2x^2-10x+14.$$

Collecting terms and dividing by -2,

$$\sqrt{x^2-3x+2}\sqrt{x^2-7x+12}=0. \quad (3)$$

Squaring and factoring,

$$(x-1)(x-2)(x-3)(x-4)=0. \quad (4)$$

Placing the factors in succession equal to zero and solving for x, $x=1, 2, 3,$ or 4. (5)

3. Solve by factoring $21x^2+4x-1=0$.

Changing the form, $$21\left(x^2+\frac{4}{21}x-\frac{1}{21}\right)=0. \quad (1)$$

Adding and subtracting $\frac{4}{441}$,

$$21\left(x^2+\frac{4}{21}x+\frac{4}{441}-\frac{4}{441}-\frac{1}{21}\right)=0. \quad (2)$$

Arranging in the form of the difference of two squares, $$21\left[\left(x+\frac{2}{21}\right)^2-\frac{25}{441}\right]=0. \quad (3)$$

Factoring,

$$21\left(x+\frac{2}{21}+\frac{5}{21}\right)\left(x+\frac{2}{21}-\frac{5}{21}\right)=0. \quad (4)$$

Collecting terms, $$21\left(x+\frac{1}{3}\right)\left(x-\frac{1}{7}\right)=0. \quad (5)$$

Placing the unknown factors in succession equal to 0 and solving for x,

$$x = -\frac{1}{3} \text{ or } \frac{1}{7}. \quad (6)$$

4. Solve by factoring $x^4 + 81 = 0$.

Adding and subtracting $18x^2$,

$$x^4 + 18x^2 + 81 - 18x^2 = 0. \quad (1)$$

Changing the form,

$$(x^2 + 9)^2 - (3x\sqrt{2})^2 = 0. \quad (2)$$

Factoring,

$$(x^2 + 3x\sqrt{2} + 9)(x^2 - 3x\sqrt{2} + 9) = 0. \quad (3)$$

Placing the factors in succession equal to 0 and solving for x,

$$x = \frac{3}{2}(\sqrt{2} \pm \sqrt{-2}) \text{ or } -\frac{3}{2}(\sqrt{2} \pm \sqrt{-2}). \quad (4)$$

EXAMPLES.

Solve by factoring:

1. $3x^2 + x = 0$.
2. $x^2 - 5x + 6 = 0$.
3. $x^4 - 2x^3 - 14x^2 + 15x + 54 = 0$.
4. $(2x - 3)(x^2 - 4) = 0$.
5. $18x^3 - 8x = 0$.
6. $(x^2 - 9)(x^2 + 3x - 10) = 0$.
7. $27x^3 - 8a^3 = 0$.
8. $x^3 - 2x^2 - x + 2 = 0$.
9. $12x^2 - 20x + 3 = 0$.
10. $(x^2 - x - 2)(x^2 + x - 2)(x^2 + 2x - 15) = 0$.
11. $x^6 - a^6 = 0$.

12. $(x^2 - 4)(2x^3 + 3x^2 - 2x - 3) = 0.$

13. $x^4 - 10a^2x^2 + 9a^4 = 0.$

14. $42 - 5x - 63x^2 = 0.$

15. $(x^2 - 4x)^2 - 2(x^2 - 4x) - 15 = 0.$

16. $(x^3 - 3x^2)^2 - (2x - 6)^2 = 0.$

17. $mnx^2 + nx - mx - 1 = 0.$

18. $x^4 + 4m^4 = 0.$

19. $\sqrt{2x - 3} - \sqrt{4 - 5x} = \sqrt{1 - 3x}.$

20. $$\frac{4}{(x+1)(x+5)} + \frac{1}{(x+1)(x+3)} = \frac{3}{(x-2)(x+4)} + \frac{2}{(x-2)(x+3)}.$$

21. $\sqrt{x + a} + \sqrt{2x - 3} = \sqrt{3x + a - 3}.$

22. $2a^2 - abx - b^2x^2 = 0.$

23. $4x^4 - 17c^2x^2 + 4c^4 = 0.$

24. $\sqrt{3 - 4x} - \sqrt{8 - 2x} = \sqrt{2x + 5}.$

25. $\sqrt{x^2 + 3x - 18} - 2\sqrt{x} = \sqrt{x^2 + 7x - 18}.$

26. $$\frac{3x^2 + 3x + 5}{x + 1} + \frac{2x^2 + 4x - 3}{x + 2} = \frac{4x^2 + 12x + 1}{x + 3} + \frac{x^2 + 4x + 1}{x + 4}.$$

27. $$\sqrt{(1 + x)(1 + 2x)} + \sqrt{(1 - x)(1 - 2x)} = \sqrt{2 + 4x^2}.$$

28. $x^4 + 5x^2 + 9 = 0.$

29. $$\left(2x - \frac{9}{2x}\right)\left(3x - \frac{4}{3x}\right) = (2x - 3)(3x - 2).$$

30. $(3-x)^3+(4-x)^3=(7-2x)^3$.
Apply Art. 90, Prin. III.

31. $a(x^3-1)=x-1$.

32. $\sqrt[3]{x-3}+\sqrt[3]{x+3}=\sqrt[3]{2x}$.

33. $x^4+4=0$.

34. $(2x-4)^3-(x+5)^3=(x-9)^3$.

IMPORTANT PROPOSITIONS REGARDING QUADRATICS.

273. The following propositions apply to all quadratic equations when reduced, as they can be, to the form $x^2+bx+c=0$.

Proposition I. *The sum of the two roots of a quadratic equation in the form $x^2+bx+c=0$, is equal to the coefficient of x with the sign changed.*

Proof. The two roots of the quadratic equation in the form given, are

$$x_1=\frac{-b+\sqrt{b^2-4c}}{2} \text{ or } -\frac{b}{2}+\frac{\sqrt{b^2-4c}}{2}$$

$$x_2=\frac{-b-\sqrt{b^2-4c}}{2} \text{ or } -\frac{b}{2}-\frac{\sqrt{b^2-4c}}{2}$$

Adding, $x_1+x_2=-\frac{2b}{2}=-b$.

Hence the proposition.

Proposition II. *The product of the two roots of a quadratic equation in the form $x^2+bx+c=0$, is equal to the third or absolute term.*

Proof. Multiplying the two roots given in Proposition I, we have

$$x_1 x_2=\frac{b^2-(b^2-4c)}{4}=\frac{4c}{4}=c.$$

Hence the proposition.

The two foregoing propositions will enable the student to obtain the sum and the product of the roots of any quadratic equation without solving it. The propositions are used quite frequently in higher mathematical work. The form of the equation to which they can be applied, should be carefully noted.

EXAMPLES.

Determine the sum and product of the roots in the following equations:—

1. $x^2+3x+5=0$.
2. $x^2-7x+2=0$.
3. $6x=9x^2+2$.
4. $2x^2-7x=8$.
5. $5-12x-6x^2=0$.
6. $5x^2-ax+bx=a^2-b^2$.

Proposition III. *Any quadratic equation in the form,* $x^2+bx+c=0$, *can, by factoring, be reduced to the form* $(x-x_1)(x-x_2)=0$, *in which* x_1 *and* x_2 *are the roots of the equation.*

The proof of this proposition is included in that of the more general proposition, in article 271. The proposition is here, however, for convenience, given a separate proof.

Proof. Representing the roots by x_1 and x_2, we have

Prop. I. $x_1+x_2=-b$, or $b=-(x_1+x_2)$.

Prop. II. $x_1x_2=c$.

Substituting these values of b and c in

$$x^2+bx+c=0,$$

we have $x^2-(x_1+x_2)x+x_1x_2=0$.

Factoring, $(x-x_1)(x-x_2)=0$.

Hence the proposition.

This proposition enables us to factor a quadratic equation of the form $x^2 + bx + c = 0$, by first finding its roots. It also enables us to form a quadratic equation which shall have any required roots.

EXAMPLES.

Reduce the following equations to factored form:

1. $x^2 - 7x + 10 = 0$.
2. $x^2 + 3x - 28 = 0$.
3. $x^2 + \frac{1}{2}x + \frac{1}{18} = 0$.
4. $x^2 - x + 1 = 0$.

Write the equations of which the following are the roots:

1. 2 and -7.
2. -1 and -6.
3. $\frac{1}{2}$ and $\frac{2}{3}$.
4. $\frac{1}{3}$ and $-\frac{1}{6}$.
5. $-\frac{a}{3}$ and $2\frac{a}{5}$.
6. $-\frac{2}{7}$ and $-\frac{3}{5}$.
7. $1 \pm \sqrt{2}$.
8. $\frac{3 \pm \sqrt{5}}{4}$.
9. $\frac{-2 \pm \sqrt{-3}}{2}$.
10. $\frac{-a \pm b\sqrt{-1}.}{2}$

Proposition IV. *Any quadratic equation in the form* $ax^2 + bx + c = 0$ *can, by factoring, be reduced to the form* $a(x - x_1)(x - x_2) = 0$, *in which* a *is the coefficient of* x^2 *and* x_1 *and* x_2 *are the roots of the equation.*

Proof. $ax^2 + bx + c = 0.$

Factoring, $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0.$

Prop. III. $a(x - x_1)(x - x_2) = 0.$

Hence the proposition.

EXAMPLES.

Reduce the following to factored form:

1. $14x^2 - 9x + 1 = 0$. 2. $12x^2 - 5x - 2 = 0$.
3. $2x^2 - 7x + 3 = 0$. 4. $6a^2x^2 + 5abx - 6b^2 = 0$.
5. $4x^2 - 4x - 1 = 0$. 6. $9x^2 - 6x + 2 = 0$.

274. By the aid of Prop. IV. any *expression*, which can be reduced to the quadratic form, can be factored. We may regard the symbol or quantity, occupying in the reduced expression the place which x occupies in the standard quadratic equation, as the only unknown quantity involved, and may consider it to have such a value as will cause the expression to equal zero. Placing the expression equal to zero, it can be reduced to the factored form by Prop. IV. The factors found, it should be noticed, are the factors of the *first member of the equation*, or, in other words, of the *expression* whose factors are sought. If we now consider this expression by itself and not as a member of an equation, the factors obtained are still the factors of the expression, *for multiplied together they will produce the expression.* They will stand this test, and therefore be correct factors, for all values of the quantitative symbols.

Suggestion.

It is generally advisable, in solving examples of this class, to resolve the factor represented by a, in the answer first obtained, into two factors which can be multiplied into the other factors in such way as to avoid fractions in the final result

Illustrative Examples.

1. Factor $42+23x-10x^2$.

The roots of the equation, $42+23x-10x^2=0$ are $-\frac{6}{5}$ and $\frac{7}{2}$.

$$\text{Therefore, } 42+23x-10x^2=-10\left(x+\frac{6}{5}\right)\left(x-\frac{7}{2}\right)$$
$$=5\left(x+\frac{6}{5}\right)(-2)\left(x-\frac{7}{2}\right)$$
$$=(5x+6)(7-2x).$$

2. Factor $2x^2y^2+xy^2+3xy-y^2+1$.

Reducing this to quadratic form and placing it equal to 0, we have

$$(2y^2)x^2+(y^2+3y)x-(y^2-1)=0.$$

The roots of this equation are $\frac{y-1}{2y}$ and $-\frac{y+1}{y}$.

The factors are, therefore,

$$2y^2\left(x-\frac{y-1}{2y}\right)\left(x+\frac{y+1}{y}\right)$$

Hence,

$$2x^2y^2+xy^2+3xy-y^2+1=(2xy-y+1)(xy+y+1).$$

EXAMPLES.

Factor the following:

1. $4x^2-2x-12$.
2. $2x^2+13x-24$.
3. $14x^2-23x+3$.
4. $5+4x-x^2$.
5. $9x^2-12x-1$.
6. $6-2x-x^2$.
7. $2x^2-4x+4$.
8. $m^2x^2-3mnx+2n^2$.
9. $1+mx-6m^2x^2$.

10. $acx^2 + (2a + c)bx + 2b^2$.

11. $15x^2 - 8ax - 2bx + a^2 - b^2$.

12. $x^2 - xy - 2x - 2y^2 + 13y - 15$.

13. $x^2 + 6x - 4z^2 + 9$.

14. $2 + y - x - 3y^2 - 4xy - x^2$.

15. $2x^2 - 7ax - x + 6a^2 + a - 1$.

16. $6a^2 - 11ab + 8a + 4b^2 - 9b + 2$.

17. $x^2 + xy + 2xz - 2y^2 - 5yz - 3z^2$.

18. $x^2yz - 2xz + 3yx - 6$.

CHAPTER XVIII.

SIMULTANEOUS EQUATIONS WITH TWO UNKNOWN QUANTITIES.

INVOLVING QUADRATICS.

275. Since the values of two unknown quantities are to be determined, two simultaneous equations are dealt with in each of the examples of this chapter. One of them must be of the second or a higher degree, in order that the solution may involve a quadratic. In certain examples the equations given are *homogeneous* or *symmetrical*.

276. Homogeneous Equations. An equation is homogeneous when the terms containing the unknown quantities are all of the same degree.

Thus, $3x^2 + 12xy + 6y^2 = 7$ is a homogeneous equation, since all the terms containing x and y are of the second degree.

277. Symmetrical Equations. An equation is symmetrical when the unknown quantities involved can be exchanged without altering the equation.

Thus, $x + y = 7$ and $2x^2 - xy + 2y^2 = 5$ are symmetrical equations, for exchanging x and y in each, we have $y + x = 7$ and $2y^2 - yx + 2x^2 = 5$, which equations are identical with those first given.

In the work of this chapter, two simultaneous equations are treated as symmetrical, if the exchange of the unknown quantities with opposite signs attached, does not alter the equations.

Thus, $x-y=10$ and $x^3-y^3=65$, are treated as symmetrical since, if in these equations, $+x$ and $-y$ be exchanged, the resulting equations $-y+x=10$ and $-y^3+x^3=65$, are identical with those first given.

278. The solution of two simultaneous quadratic equations will, in general, lead to an equation of the fourth degree.

Given, for instance, the two simultaneous equations,

$$x^2+xy+y^2=19. \qquad (1)$$

$$x^2+y=7. \qquad (2)$$

Solving (2) for y, $$y=7-x^2. \qquad (3)$$

Substituting in (1), $$x^2+x(7-x^2)+(7-x^2)^2=19. \qquad (4)$$

Form 4, $$x^4-x^3-13x^2+7x+30=0. \qquad (5)$$

It is only in a limited number of examples that the elimination of one of the unknown quantities will furnish an equation, which can be solved by any of the methods so far discovered.

279. Attention is called to the following important proposition:—

Proposition. *If two equations, of the first and second degree, respectively, involving two unknown quantities, are combined and one of the unknown quantities eliminated, the resulting equation will be a quadratic.*

Proof. The general equation of the first degree involving two unknown quantities is,

$$a_1x+b_1y+c_1=0. \qquad (1)$$

Solving for y, $y=\dfrac{-a_1x-c_1}{b_1}=-\dfrac{a_1}{b_1}x-\dfrac{c_1}{b_1}.$

The general equation of the second degree involving the same unknown quantities is,

$$ax^2+bxy+cy^2+dy+ex+f=0. \quad (2)$$

Combining the two equations by substituting in equation (2), the value of y, in terms of x, derived from equation (1), it is evident that the highest power of x in the resulting equation will be its square.

Hence the proposition.

We have then the following rule for solving two simultaneous equations, when one is of the first, and the other of the second degree.

RULE.

1. *Find the value of one of the unknown quantities in terms of the other from the equation of the first degree, and substitute this value in the other equation.*

2. *Solve the resulting quadratic.*

3 [illegible] obtained [illegible] first degree [illegible]

EXAMPLES.

1. $\begin{cases} x^2+y^2=10. \\ x+y=4. \end{cases}$

2. $\begin{cases} x-y=3. \\ xy=10. \end{cases}$

3. $\begin{cases} 8x+2y=18. \\ x^2-xy=2. \end{cases}$

4. $\begin{cases} x^2-2y^2=7. \\ 2x+y=13. \end{cases}$

5. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{4}=\dfrac{25}{12}. \\ \dfrac{4}{x}+\dfrac{3}{y}=2. \end{cases}$

6. $\begin{cases} x^2 - xy - y^2 = 4. \\ x + 2y = 8. \end{cases}$

7. $\begin{cases} \dfrac{x}{y} - \dfrac{2y}{x} = 1. \\ 3y - x = 4. \end{cases}$

280. In the solution of simultaneous quadratics, the importance of the proposition proven in the preceding article, cannot be too strongly emphasized. If, in any example, the equations given can be so handled as to furnish two equations, one of the first and the other of the second degree, a ready solution from there on is guaranteed by the proposition.

The whole thought of the student, then, should be devoted to obtaining these two equations. The criterion of success in any particular example is the discovery of the combinations or steps by which these equations can be found. Experience will reveal special ways in special examples.

The following methods of procedure are suggested for the classes of examples considered:

281. Case I. *When both of the equations are of the second degree, and one is homogeneous with its absolute term zero.*

Treating the homogenous equation as an affected quadratic, solve for either unknown quantity. Separating the double sign, two equations will result, each of the first degree; while the other given equation is of the second degree. Two examples will thus be formed, each fitting under the proposition. In solving, they should be treated separately. Four pairs of values of x and y will result.

Illustrative Example.

Solve the equation $\begin{cases} (1)\ x^2 - xy - 2y^2 = 0, \\ (2)\ 3x^2 + 5x - 7y^2 = 15. \end{cases}$

Factoring, (1) $(x - 2y)(x + y) = 0,$

Whence, (3) $x = 2y,$

(4) $x = -y$

Substituting from (3) in (2),

$$3(2y)^2 + 5(2y) - 7y^2 = 15$$

Or $5y^2 + 10y = 15,$

Which solved gives, $y = 1 \text{ or } -3,$

$x = 2 \text{ or } -6.$

Substituting from (4) in (2),

$$3y^2 - 5y - 7y^2 = 15$$

Or $-4y^2 - 5y = 15$

Which solved gives, $y = \dfrac{-5 \pm \sqrt{-215}}{8},$

$x = \dfrac{5 \mp \sqrt{-215}}{8}.$

EXAMPLES.

Solve the following:

1. $\begin{cases} x^2 + xy = 0, \\ 17x^2 + y^2 + 2y = 440. \end{cases}$

2. $\begin{cases} xy - y^2 = 0, \\ 2x^2 + xy - 5y = 2. \end{cases}$

3. $\begin{cases} x^2 + 2xy - 8y^2 = 0, \\ x^2 + 2x - 3y = 5. \end{cases}$

4. $\begin{cases} y^2 + xy - 20x^2 = 0, \\ x^2 + y^2 + x + y = 22. \end{cases}$

282. Case II. *When both of the equations are of the second degree and homogeneous.*

Combine the two equations, eliminating the absolute term. The resulting equation will be homogeneous, with its absolute term zero. Taken with either of the two given equations, it will form an example which can be solved under Case I.

Remarks on Examples.

1. The derived homogeneous equation should be solved in connection with the simpler of the given equations, and that only; if solved in connection with the other, no new values of x and y will be found.

2. As in Case I, the solution will develop two subordinate examples, which should be dealt with separately. The values of x and y found, should be carefully assorted into pairs.

3. Examples which are both quadratic and homogeneous with reference to any powers of x and y, or with reference to any expressions of the first degree involving x and y, can be solved by the method employed in this case; for it is evident that such quantities or expressions can be represented by single letters, and the method of this case then used to find their values. This done, values of x and y can readily be determined. (See second illustrative example.)

Illustrative Examples.

1. Solve the equations $\begin{cases} (1)\ x^2 + xy = 21, \\ (2)\ y^2 - xy = 4. \end{cases}$

Multiplying (1) by 4
and (2) by 21, $4x^2 + 4xy = 84.$ (3)

$\underline{21y^2 - 21xy = 84.}$ (4)

Subtracting (4) from (3), $4x^2 + 25xy - 21y^2 = 0.$ (5)

Solving (5) for x, $x = \frac{3}{4}y$, (6)

Or $x = -7y$. (7)

Substituting from (6) in (2), $y^2 - \frac{3}{4}y^2 = 4$, (8)

Whence, $y = \pm 4$, (9)

$x = \pm 3$. (10)

Substituting from (7) in (2), $y^2 + 7y^2 = 4$, (11)

Whence, $y = \pm \frac{1}{2}\sqrt{2}$, (12)

$x = \mp \frac{7}{2}\sqrt{2}$. (13)

2. Solve the equations

$$\begin{cases} (x+y+1)^2 + x^3(x+y+1) + x^6 = 21, & (1) \\ (x+y+1)^2 - x^3(x+y+1) = 12. & (2) \end{cases}$$

Multiplying (1) by 4 and (2) by 7,

$$4(x+y+1)^2 + 4x^3(x+y+1) + 4x^6 = 84, \quad (3)$$

$$7(x+y+1)^2 - 7x^3(x+y+1) = 84. \quad (4)$$

Subtracting (4) from (3),

$$-3(x+y+1)^2 + 11x^3(x+y+1) + 4x^6 = 0. \quad (5)$$

Solving (5) for $x+y+1$, $x+y+1 = 4x^3$, (6)

Or $x+y+1 = -\frac{1}{3}x^3$. (7)

Substituting in (2) from (6), $16x^6 - 4x^6 = 12$. (8)

$x^6 = 1$. (9)

$x = 1$ or -1. (10)

$y = 2$ or -4. (11)

Substituting in (2) from (7), $\frac{1}{9}x^6 + \frac{1}{3}x^6 = 12$. (12)

$x = \pm\sqrt{3}$. (13)

$y = \mp 2\sqrt{3} - 1$. (14)

Solve the following equations:

1. $$\left\{\begin{array}{l} 2x^2 + y^2 = 9, \\ xy - x^2 = -2. \end{array}\right.$$

2. $$\left\{\begin{array}{l} x^2 - 3xy = -20, \\ y^2 - 3x^2 = 4. \end{array}\right.$$

3. $$\left\{\begin{array}{r} x^2 + xy + y^2 = 19, \\ 2x^2 + y^2 = 22. \end{array}\right.$$

4. $$\left\{\begin{array}{r} x^2 + y^2 - xy = 3, \\ x^2 - 2yx + 3y^2 = 9. \end{array}\right.$$

5. $$\left\{\begin{array}{r} (x+1)^2 + (x+1)(y+1) + (y+1)^2 = 7. \\ 2(x+1)^2 - 3(y+1)^2 = 5. \end{array}\right.$$

6. $$\left\{\begin{array}{r} x^4 - 3x^2y^3 + y^6 = 5, \\ x^4 + 6x^2y^3 = 40. \end{array}\right.$$

283. Case III. *When both equations are symmetrical with reference to x and y.*

First Method. Find values of $x + y$, or $x - y$, for equation of the first degree. In the process, an equation of the second degree will generally be evolved, if one is not already given in the example. Then solve under the proposition.

Second Method. An elegant method of solving examples of this class is to find values of both $x + y$ and $x - y$. While the proposition under which we are working, requires that only one of two simultaneous equations shall be of the first degree, in order to insure a solution, we have here a case in which, in the subordinate examples developed by our combinations, the equations are both of the first degree. This makes the work of solution so much the easier and more elegant.

Remarks on Examples.

1. In seeking the value of $x+y$ or $x-y$, the aim of the student, in general, should be so to combine the given equations as to obtain an equation whose first member is an exact power of $x+y$ or $x-y$. The values sought can then be found by extracting the proper root of each member of the equation. In certain examples a single step, such as dividing one equation by the other, will give the values desired. It is sometimes convenient, in the process of the work, to solve with reference to xy or $\frac{x}{y}$.

2. When values are found for both $x+y$ and $x-y$, they will generally be of the form

$$1.\quad x+y=\pm a,$$
$$2.\quad x-y=\pm b.$$

These when properly separated will give for solution, four pairs of simultaneous equations:

$$\begin{array}{llll} 1. & x+y=a, & & x-y=b. \\ 2. & x+y=a, & & x-y=-b. \\ 3. & x+y=-a, & & x-y=b. \\ 4. & x+y=-a, & & x-y=-b. \end{array}$$

3. The value of $x+y$ or $x-y$ may be given in one of the original equations. The example can then be solved by substitution or by the second method.

4. Simultaneous equations, symmetrical with reference to x and y, may be solved by substituting $u+v$ and $u-v$ for x and y. The other methods suggested are to be preferred.

5. The methods of case III can be applied to examples, in which the equations are symmetrical with reference to any powers of x and y, or any powers of expressions of the first degree involving x and y. (See third illustrative example.)

Illustrative Examples.

1. Solve the equations, $\begin{cases} x^2 + y^2 = 29, & (1) \\ xy = 10. & (2) \end{cases}$

Multiplying (2) by 2, $2xy = 20.$ (3)

Adding (1) and (3),

$$x^2 + 2xy + y^2 = 49. \quad (4)$$

Subtracting (3) from (1),

$$x^2 - 2xy + y^2 = 9. \quad (5)$$

Extracting the square root of (4) and (5),

$$x + y = \pm 7, \quad (6)$$

$$x - y = \pm 3. \quad (7)$$

Adding (6) and (7), $x = \pm 5$ or $\pm 2.$ (8)

Subtracting (7) from (6), $y = \pm 2$ or $\pm 5.$ (9)

2. Solve the equations, $\begin{cases} x^4 + y^4 = 17, & (1) \\ x + y = 3. & (2) \end{cases}$

Raising (2) to the 4th power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81. \quad (3)$$

Subtracting (1) from (3),

$$4x^3y + 6x^2y^2 + 4xy^3 = 64. \quad (4)$$

Squaring (2) and multiplying by $4xy$,

$$4x^3y + 8x^2y^2 + 4xy^3 = 36xy. \quad (5)$$

Subtracting (4) from (5), $2x^2y^2 = 36xy - 64.$ (6)

Solving for xy, $xy = 16$ or $2.$ (7)

The solution from this point is similar to example 1.

3. Solve the equations

$$\begin{cases} (x-y+3)^3+y^6=65, & (1) \\ (x-y+3)^2-y^2(x-y+3)+y^4=13. & (2) \end{cases}$$

Dividing (1) by (2), $(x-y+3)+y^2=5.$ $\quad(3)$

Squaring (3),

$$(x-y+3)^2+2(x-y+3)y^2+y^4=25. \qquad (4)$$

Subtracting (2) from (4), $3(x-y+3)y^2=12.$ $\quad(5)$

$$(x-y+3)y^2=4. \qquad (6)$$

Subtracting (6) from (2),

$$(x-y+3)^2-2(x-y+3)y+y^4=9. \qquad (7)$$

Extracting square root of (7),

$$(x-y+3)-y^2=\pm 3. \qquad (8)$$

Adding (8) and (3), $x-y+3=4 \text{ or } 1.$ $\quad(9)$

Subtracting (8) from (3), $y^2=1 \text{ or } 4.$ $\quad(10)$

Whence, $y=1, -1, 2 \text{ or } -2,$ $\quad(11)$

$x=2, \quad 0, 0 \text{ or } -4.$ $\quad(12)$

EXAMPLES.

Solve the equations:

1. $\begin{cases} x^2+y^2=25, \\ x+y=7. \end{cases}$

2. $\begin{cases} x^2+y^2=52, \\ xy=24. \end{cases}$

3 $\begin{cases} x^3+y^3=28, \\ x+y=4. \end{cases}$

4. $\begin{cases} x^3+y^3=65, \\ x^2y+xy^2=20. \end{cases}$

5. $\begin{cases} x^6-y^9=63, \\ x^4+x^2y^3+y^6=21. \end{cases}$

6. $\begin{cases} (x+y)^2+(x-y)^2=10, \\ (x+y)(x-y)=3. \end{cases}$

284. Case IV. *When, from the given equations, an equation can be obtained, whose second member is known, and whose first member is an exact power of any expression of the first degree, involving x and y.*

Extract the proper root of each member of the derived equation. The required equation of the first degree will thus be obtained.

Remarks on Examples.

1. It will often occur that one of the given equations, or a derived equation, can be put in quadratic form, with reference to the expression. The value of the expression can then be found by completing the square, or factoring. The latter method is, in general, the more elegant. (See illustrative example.)

2. The method of this *case* is evidently applicable to equations above the second degree.

Illustrative Example.

1. Solve the equations, $\begin{cases} x^2+xy+2y=7, & (1) \\ y^2+16x=20. & (2) \end{cases}$

Multiply (1) by 4, $4x^2+4xy+8y=28.$ (3)

Adding (2) and (3),

$$4x^2+4xy+y^2+8(2x+y)=48. \quad (4)$$

Changing the form of (4),

$$(2x+y)^2+8(2x+y)-48=0. \quad (5)$$

This is in quadratic form with reference to $2x+y$, hence the value of $2x+y$ may be found either by completing the square, by the formula, or by factoring.

Factoring (5),

$$(2x+y+12)(2x+y-4)=0, \quad (6)$$

Whence, $y=-12-2x$ (7)

Or, $y=4-2x.$ (8)

Substituting in (1),

$$x^2-12x-2x^2-24-4x=7. \quad (9)$$

Or, $x^2 + 4x - 2x^2 + 8 - 4x = 7$. (10)

Whence, $x = -8 \pm \sqrt{33},\ 1 \text{ or } -1,$ (11)

$y = 4 \mp 2\sqrt{33},\ 2 \text{ or } 6.$ (12)

EXAMPLES.

Solve the following:

1. $\begin{cases} x^2 + y^2 + 2(x + y) = 11, \\ xy = 2. \end{cases}$

2. $\begin{cases} x^2 + xy = 3, \\ y^2 + xy = 6. \end{cases}$

3. $\begin{cases} x^2 - 3xy + 4y^2 - 7x = -12, \\ 5y^2 - 3xy + 21y = 20. \end{cases}$

4. $\begin{cases} (x^2 + y)^2 + y = 11, \\ 4x^2 + 3y = 10. \end{cases}$

285. Remarks on Miscellaneous Examples.

The student should remember that the most important step in the solution of these simultaneous equations, is to bring the example under the fundamental proposition. The greatest difficulty to be overcome will, in general, be the obtaining of an equation of the first degree. Other devices than those already mentioned, for this purpose, can often be employed to advantage. Sometimes the terms of higher degree can be eliminated from the two given equations by the process of addition and subtraction; sometimes one equation can be divided by the other and the elimination thereby of a factor common to the first members of the two, result in an equation of lower degree; *whatever* combination is made, should be directed to the purpose of obtaining *two equations, neither of which, with* *reference to the unknown quantities or expressions involved, shall be of a higher degree than the second, and one of which, at least, shall be of the first degree, with reference to such quantities or expressions.*

MISCELLANEOUS EXAMPLES.

Solve the following equations:

1. $\begin{cases} x - 2y = 3, \\ xy = 14. \end{cases}$

2. $\begin{cases} x^2 + xy = 15, \\ xy + y^2 = 10. \end{cases}$

3. $\begin{cases} x^3 - 125y^3 = 91, \\ x - 5y = 1. \end{cases}$

4. $\begin{cases} x^2 + 2xy = 35, \\ 3x - 4y = 11. \end{cases}$

5. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2}, \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{5}{4}. \end{cases}$

6. $\begin{cases} x^2y^2 - 18xy + 56 = 0, \\ x + y = 4. \end{cases}$

7. $\begin{cases} 9x^2 - 5xy = 21, \\ xy - y^2 = -4. \end{cases}$

8. $\begin{cases} x^4 + y^4 = 626, \\ x + y = 6. \end{cases}$

9. $\begin{cases} x^3 - y^3 = \dfrac{19xy}{6}, \\ x - y = 1. \end{cases}$

10. $\begin{cases} \dfrac{1}{x^3} - \dfrac{1}{y^3} = \dfrac{7}{8}, \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{2}. \end{cases}$

11. $\begin{cases} xy + y^2 = 7, \\ x^2 - y^2 = 35. \end{cases}$

12. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = 9. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{4}. \end{cases}$

13. $\begin{cases} x^2 + 9y^2 = 5x + 15y, \\ xy = 6. \end{cases}$

14. $\begin{cases} x^2 + 8y^2 = 12. \\ xy + y^2 = 3. \end{cases}$

15. $\begin{cases} x^2 + y = x, \\ y^2 + x = y. \end{cases}$

16. $\begin{cases} xy = (x - y) + 13, \\ x^2 + y^2 = 34. \end{cases}$

17. $\begin{cases} x^2+y^2=\dfrac{10xy}{3}, \\ x+y=\dfrac{2xy}{3}. \end{cases}$

18. $\begin{cases} x^5+y^5=275, \\ x+y=5. \end{cases}$

19. $\begin{cases} x^2-xy=2b\,(b-a), \\ xy-y^2=-2b\,(b+a). \end{cases}$

20. $\begin{cases} 9x^2+25y^2-6x=163, \\ 30xy-10y=160. \end{cases}$

21. $\begin{cases} x+\sqrt{xy}+y=19, \\ 2x-3\sqrt{xy}+y=-1. \end{cases}$

22. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\sqrt{\dfrac{1}{x}+\dfrac{1}{y}}=2, \\ \dfrac{1}{xy}=3 \end{cases}$

23. $\begin{cases} 9x^2+4xy+y^2=45, \\ 7x^2+12xy+2y^2=54. \end{cases}$

24. $\begin{cases} x^2-2xy-y+5x=10. \\ y^2+x-5y=-3. \end{cases}$

25. $\begin{cases} x^4+x^2y^2+y^4=21, \\ x^2+xy+y^2=7. \end{cases}$

26. $\begin{cases} 9x^2+5xy+2y^2=7, \\ 7x^2+10xy=-13. \end{cases}$

27. $\begin{cases} x^4+y^4=85, \\ x^2+\sqrt{2}xy+y^2=17. \end{cases}$

28. $\begin{cases} x^2y^2+\dfrac{1}{x^2y^2}-2\left(xy+\dfrac{1}{xy}\right)=-\dfrac{3}{4}, \\ x+y=3. \end{cases}$

29. $\begin{cases} x^2+2xy=ax+by, \\ y^2=bx+ay. \end{cases}$

30. $\begin{cases} x^2+y^2+\sqrt{x^2+y^2+3}=9. \\ xy=6. \end{cases}$

31. $\begin{cases} x^2+xy-3y^2=-9, \\ 5x^2-7xy+4y^2=18. \end{cases}$

32. $\begin{cases} x^2+xy+y^2-2\sqrt{x^2+xy+y^2}=3, \\ x^3-y^3=54. \end{cases}$

33. $\begin{cases} x-y-4\sqrt{x-y}=-3, \\ x^2+y^2+2\sqrt{x^2+y^2}=35. \end{cases}$

34. $\begin{cases} \sqrt{\dfrac{x+y}{y}}-6\sqrt{\dfrac{y}{x+y}}+1=0, \\ x^2+2y=11. \end{cases}$

35. $\begin{cases} x^3-3x^2y+3xy^2-9y^3=0, \\ xy=3. \end{cases}$

36. $\begin{cases} x+y+4\sqrt{\dfrac{x+y}{x-y}}=\dfrac{21}{x-y}. \\ 2x+y=14. \end{cases}$

37. $\begin{cases} x+y=9, \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=3. \end{cases}$

38. $\begin{cases} xy+x+y=7, \\ x^2y+xy^2=12. \end{cases}$

39. $\begin{cases} x^2-y^2+x-y=4, \\ (x-y)(x+2y-3)=1. \end{cases}$

40. $\begin{cases} \sqrt{x-y}+xy=3, \\ xy\sqrt{x-y}=2. \end{cases}$

41. $\begin{cases} (2x+3y-4)(x-y)+(x-y)^2=4, \\ (2x+3y-4)^2+(x-y)^2=10. \end{cases}$

PROBLEMS LEADING TO SIMULTANEOUS QUADRATIC EQUATIONS.

286. 1. The sum of the squares of two numbers is 34, and their product is 15. What are the numbers?

2. There are two numbers, such that the square of the first exceeds the product of the two by 10, and four times the square of the second plus 5 times the product of the two, equals 111. What are the numbers?

3. A number has two digits. The product of the number by the number formed by reversing the digits is 252, and the product of the digits is 2. What is the number?

4. A fraction is such that if the numerator is increased by 11 and the denominator by 1, it will then equal its reciprocal. The square of the denominator exceeds 5 times the numerator by an amount equal to 1 more than the product of the numerator and denominator. Find the fraction.

5. A sum of $2,500 at simple interest yields $600. If the rate had been 2% higher, the same interest would have accrued in 2 years less time. Find the time and rate.

6. The area of a garden in rectangular form is 10 acres. The fence around it is 200 rods in length. Find the dimensions of the garden.

7. A fruit dealer invested $3 in apples and found that a rise of $\frac{1}{4}$ cent per apple, enabled him to sell 120 less than $\frac{3}{4}$ of his apples for the amount originally invested. How many did he buy, and what was the cost price of each apple?

8. Two men, riding wheels, start toward each other from two towns 140 miles apart. When they meet, it is found that A could have gone B's distance in 6

hours, and B could have gone A's in $10\frac{2}{3}$ hours. How far had each traveled, at what rate, and when did they meet?

9. The sum of two numbers is x and the sum of their squares is y. Find the numbers.

10. A starts from M to N; and three hours later B starts, traveling at a rate 2 miles an hour faster than A. C, starting 3 hours later than B, and traveling at a rate of 12 miles an hour, reaches the station N at the same time as A and B. What is the distance from M to N, and the rate and time of each?

11. The hind wheel of a wagon exceeds the fore-wheel in circumference by 16 inches, and makes 44 less revolutions in each mile it goes. What is the circumference of each?

12. I bought bonds at as many per cent above par as the rate of interest which they bore, and sold them, after a year, at one and one-half times as many per cent, above par, as this rate of interest. My income from this investment was a per cent of the amount invested equal to one and ten thirteenths per cent more than the rate of interest on the bonds. Required the rate, and the price paid for the bonds.

13. In a garden, the number of rods in the perimeter equals the number of square rods in the area. If the length were increased by 4 rods and the breadth by 6, the area would be one-half an acre. What is the area of the garden?

14. Two men rented a house, A promising to pay, as his share, $150 a year. After six months B, having a less number of rooms than A, decided to take

two of the rooms rented by A; and at the end of the year his rent cost him two and one-half times the difference of the two annual rents, they had first agreed to pay. The number of rooms in the house was one-tenth the number of dollars which B paid per year for the rooms he first engaged. What was the rent per room, and the number of rooms each engaged at first?

15. On a piece of work, two men are engaged for a certain number of days at different rates per day. At the end, they receive \$60, the first having been idle 5, and the second 10 days. Had the first been idle 10, and the second 5 days, they would have received \$65. The two together receive one-third as many dollars in a day as the number of days the first worked. How many days did each work, and at what rates per day?

16. Two trains starting at the same time and going in opposite directions between M and N, upon meeting, have differed in the distance covered by 20 miles. It is found that the train from M will reach N in one and one-half hours from the time of meeting, and the train from N will reach M in 40 minutes. How far apart are M and N, and what is the rate of each train?

17. The wheels of two bicycles having tires of different sizes, make the same number of revolutions in a second. The rider of the wheel with the large tire, finds that he can go 2 miles an hour faster than the other. The difference in the number of revolutions, in going a mile, is 528. If the wheels made $\frac{13}{15}$ of a revolution less per second, the number of revolutions per second would be $\frac{1}{2}$ the number of miles

the larger wheel goes in one hour. What is the number of revolutions of each per second, and the rate of each in miles per hour?

18. Two boys in running to a certain post and back again, find there is a difference of 5 seconds in the time required for the race. There is a difference of 5 yards per second in the rates of their running, and if the slower had turned back when he met the faster, there would have been a difference of but $2\frac{1}{7}$ seconds at the end of the race. How many yards can each run in a second, and how far is the post from the starting point?

19. Two boys, A and B, agreed to move a quantity of wheat from one place to another. A can carry 10 pounds more than B and together they carry all the wheat in 12 trips each. If the wheat had been divided into two equal parts, and each had carried half, it would have required 25 trips in all. How many pounds of wheat did each carry, and how many pounds were to be moved?

20. Two men invested different sums of money. A withdraws his in 6 months and B in 10. If B had invested $50 less, the rates of profit received by A and B respectively, would have been as 9:5. A's capital and profit when taken out amount to $318 and B's to $165. What did each invest?

CHAPTER XIX.

PERMUTATIONS AND COMBINATIONS.

PERMUTATIONS.

287. Permutations are the different orders in which things can be arranged. Thus, *a* and *b* can be written *ab* and *ba*; *a*, *b* and *c* can be written *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

288. In forming permutations, it is not necessary that all of the things given should appear in each permutation. Three things, for instance, may be given and the example specify that each permutation shall include only two of them. A base ball manager may have 12 men at his disposal; the example in permutations, which he has to consider, is to determine the different orders in which he can place 12 men, taking 9 at a time. In the general example, we are asked to find the number of permutations which can be formed with n things taken r at a time. ${}_nP_r$ is usually employed to symbolize what is required in this case; the symbol may be used in any particular example by substituting the proper values for n and r.

289. We may reach an understanding of the solution of the general example by considering a series of preliminary examples leading to it in logical order.

Preliminary Examples.

1. Find the number of permutations of n different things, taken 2 at a time.

Solution. Let the things given be represented by $a_1, a_2, a_3, \ldots\ldots\ldots a_n$. Take a_1, and write after it, in succession, each of the other letters, in the order in which they are given; $n-1$ permutations will thus be formed. Treating each letter in the same way and arranging the permutations formed, in columns, for convenience, we have,

$$\begin{array}{c|c|c|c|c|}
a_1a_2 & a_2a_1 & a_3a_1 & \ldots\ldots & a_na_1 \\
a_1a_3 & a_2a_3 & a_3a_2 & \ldots\ldots & a_na_2 \\
a_1a_4 & a_2a_4 & a_3a_4 & \ldots\ldots & a_na_3 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
a_1a_n & a_2a_n & a_3a_n & \ldots\ldots & a_na_{n-1}
\end{array}$$

Since there are n letters, there are n columns, and there are $n-1$ permutations in each column. No other permutations, taken two at a time, can be formed.

Hence, ${}_nP_2 = n\,(n-1)$.

2. Find the number of permutations of n different things taken three at a time.

Solution. Investigation will show that all possible permutations taking three letters at a time can be obtained by writing in succession, after each permutation containing two letters, the remaining letters given. Writing after a_1a_2, in succession, $a_3, a_4 \ldots\ldots a_n$, $n-2$ permutations will be formed. In a similar way, an equal number of permutations, containing three letters, can be made in connection with each permutation of two letters. Arranging these in columns, we have,

$$\begin{array}{c|c|c|c|c|}
a_1a_2a_3 & a_1a_3a_2 & a_1a_4a_2 & \cdots\cdots & a_na_{n-1}a_1 \\
a_1a_2a_4 & a_1a_3a_4 & a_1a_4a_3 & \cdots\cdots & a_na_{n-1}a_2 \\
a_1a_2a_5 & a_1a_3a_5 & a_1a_4a_5 & \cdots\cdots & a_na_{n-1}a_3 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots \\
a_1a_2a_n & a_1a_3a_n & a_1a_4a_n & \cdots\cdots & a_na_{n-1}a_{n-2}
\end{array}$$

There are $n-2$ permutations in each column, and there are as many columns as there are permutations of n things taken 2 at a time, which we have found to be $n\,(n-1)$.

Hence, ${}_nP_3 = n\,(n-1)\,(n-2)$.

3. Find the number of permutations of n different things taken 4 at a time.

Solution. It is evident that if we write after each permutation of three letters, the other given letters, in succession, we will have all the permutations which can be formed with 4 things in a permutation.

Hence, ${}_nP_4 = n\,(n-1)\,(n-2)\,(n-3)$.

General Example.

Find the number of permutations of n different things taken r at a time.

Solution. The answers to our preliminary examples give us,

$$\begin{aligned}
{}_nP_2 &= n\,(n-1), \\
{}_nP_3 &= n\,(n-1)\,(n-2), \\
{}_nP_4 &= n\,(n-1)\,(n-2)\,(n-3).
\end{aligned}$$

From the method employed in finding the number of permutations in each case, it is evident that if the work were continued, we would have,

$${}_nP_5 = n(n-1)(n-2)(n-3)(n-4),$$
$${}_nP_6 = n(n-1)(n-2)(n-3)(n-4)(n-5),$$

and so on.

It will be noted that the factors in each example are continuous from n down to a factor, in which the number subtracted from n is one less than the number of letters taken in each permutation. Since this grows out of the nature of the work and is consequently true when *any* number of letters are taken in a permutation, we have the

General Formula.

$${}_nP_r = n(n-1)(n-2)(n-3)\ldots\ldots(n-[r-1]),$$

which reduces to

$${}_nP_r = n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1). \qquad (1)$$

290. Corollary. When all the things given are included in each permutation $n = r$, and the formula becomes,

$${}_nP_n = n(n-1)(n-2)(n-3)\ldots\ldots\ldots 1,$$

or, writing the factors in a reverse order,

$${}_nP_n = 1.2.3.4.5.6\ldots n = \underline{|n}.^* \qquad (2)$$

291. Any particular example can be solved by making the proper substitutions in the general formulas.

EXAMPLES.

1. How many different numbers of three figures each can be formed from the figures, 1, 2, 3, 4, 5, 6, and 7?

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* NOTE. The symbol $\underline{|n}$, read factorial n, signifies the product of the natural numbers from 1 to n inclusive.

Putting $n = 7$ and $r = 3$ in (1)

$$_7P_3 = 7.6.5. = 210.$$

2. Find the value of $_{10}P_3$; $_6P_5$; $_{21}P_5$.

3. How many words can be made from the 26 letters of the alphabet using 4 at a time?

4. In how many different ways can 6 people be seated at a dinner table?

5. How many different words can be made from the letters in the word *drawing*, each word containing 7 letters?

COMBINATIONS.

292. Combinations of things are the different groups or collections which can be formed, without reference to the order in which the things are arranged.

293. The difference between *combinations* and *permutations* should be carefully noted. Two permutations (ab and ba) can be made with the letters a and b, but only one combination. The letters a, b and c, when all are taken at a time, can be arranged, as we have seen, in six different orders, making six permutations; but in whatever order they are placed, they constitute one and the same combination. No change of order within a group will make a new group; it is only by an interchange of things between groups, that a new combination will be formed.

294. Combinations, like permutations, need not contain all of the things given. We may be asked, for instance, to find the number of combinations of 6 things taken 5 at a time, and, in the most general example, of n things taken r at a time. What is required in this latter case is symbolized by ${}_nC_r$, in which n denotes the number of things given, and r the number taken in each combination.

295. General Example.

Find the number of combinations of n different things taken r at a time.

Solution. We can best solve this example through a study of the relation of permutations to combinations. Taking first a particular case, let it be required to find the number of different combinations or groups which can be formed of 20 students placing 5 in a group. We can first separate them into 4 groups. Any interchange of students between groups will make a new group. Proceeding in this way, we will at length have made all the groups or combinations possible. We may now conceive that in each group, the students are arranged in different orders, and all possible permutations with five in a permutation formed. There will be the same number of permutations in each group and no other permutations of five can be made. We will then have the following equation:

No. of permutations = no. of combinations × no. of permutations in each combination or group, when all in the group are taken.

This equation is evidently true, whatever the number of things given, or the number included in each combination. The general equation may, therefore, be written thus,

$${}_nP_r = {}_nC_r \times \underline{|r}.$$

Solving for ${}_nC_r$, we have the answer to the general example which will constitute a

General Formula.

$$ {}_nC_r = \frac{{}_nP_r}{\underline{|r}} $$

or $$ {}_nC_r = \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{1.2.3.4.5\ldots\ldots r}. \quad (3) $$

Corollary. By substituting in succession 1, 2, 3 and 4 for r in (3), we have,

$$ {}_nC_1 = n. \quad (4) $$

$$ {}_nC_2 = \frac{n(n-1)}{1.2}. \quad (5) $$

$$ {}_nC_3 = \frac{n(n-1)(n-2)}{1.2.3}. \quad (6) $$

$$ {}_nC_4 = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}. \quad (7) $$

296. Particular examples can be solved by making proper substitutions in the general formula.

EXAMPLES.

1. How many different committees of three can be made from a society of 20 persons?

Putting $n = 20$ and $r = 3$ in (3),

$$ {}_{20}C_3 = \frac{20.19.18}{1.2.3} = 1140. $$

2. Find the value of ${}_{15}C_5$; ${}_{19}C_7$; ${}_{20}C_{15}$.

3. On a tennis court, where but 4 people can play, are 6 players. In how many ways can the set be made up?

4. How many sums can be made from the figures, 1, 2, 3, 4, 5, and 6, using 3 at a time?

5. A hostess has a company of 20 people to place for supper at small tables, at each of which 4 can be seated. In making up the possible groups of 4, how many such combinations can she form?

297. The following proposition will be used in the proof of the binomial theorem:

Proposition. *The number of combinations of n things taken r at a time, plus the number of combinations of n things taken r + 1 at a time, is equal to the number of combinations of n + 1 things taken r + 1 at a time.*

Proof.

From art. 295, we have,

$$ {}_nC_r = \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{1.2.3.4.5\ldots.r}. \quad (1)$$

If $r + 1$ things are taken in each combination, we will have by substituting $r + 1$ for r in formula (1),

$$ {}_nC_{r+1} = \frac{n(n-1)\ (n-2)\ (n-3)\ldots.(n-r+1)}{1.2.3.4\ldots.r}\cdot\frac{n-r}{r+1}. \quad (2)$$

If we find the number of combinations of $n + 1$ things, taken $r + 1$ at a time, we will have, by substituting in formula (1),

$$ {}_{n+1}C_{r+1} = \frac{(n+1)\ n(n-1)\ (n-2)\ (n-3)\ \ldots(n-r+1)}{1.2.3.4\ldots.r\ (r+1)}. \quad (3)$$

Representing the second member of (1) by K and substituting K in (2) and (3), we may re-write (1), (2), and (3), thus,

$$ {}_nC_r = K. \quad (4)$$

$$ {}_nC_{r+1} = K\,\frac{n-r}{r+1}. \quad (5)$$

$$ {}_{n+1}C_{r+1} = K\,\frac{n+1}{r+1}. \quad (6)$$

Adding (4) and (5),

$$ {}_nC_r + {}_nC_{r+1} = K + K\frac{n-r}{r+1} = K\frac{r+1+n-r}{r+1} = K\frac{n+1}{r+1}. \quad (7)$$

Comparing with (6),

$$ {}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}. \quad (8)$$

Hence the proposition.

Corollary. By substituting in succession 1, 2, 3 and 4 for r in (8), we have,

$$ {}_nC_1 + {}_nC_2 = {}_{n+1}C_2. \quad (9)$$

$$ {}_nC_2 + {}_nC_3 = {}_{n+1}C_3. \quad (10)$$

$$ {}_nC_3 + {}_nC_4 = {}_{n+1}C_4. \quad (11)$$

PROMISCUOUS EXAMPLES.

298. Illustrative Example.

1. How many words of 3 consonants and 2 vowels, can be made from the letters, a, b, c, d, e, m, n, o?

Solution. The number of combinations of 5 consonants, taken 3 at a time is $\frac{5.4.3}{1.2.3} = 10$.

The number of combinations of 3 vowels taken 2 at a time is $\frac{3.2}{1.2} = 3$.

Each group of consonants may be put with each group of vowels, thus making in all 30 sets of letters, each consisting of 2 vowels and 3 consonants. But any one of these sets of 5 letters can be arranged in as many ways as there are permutations of 5 taken 5 at a time, or $\underline{|5} = 120$.

Therefore there are $30 \times 120 = 3{,}600$ words of 2 vowels and 3 consonants.

2. How many different sums of money can be made from a penny, a nickel, a dime, a quarter, a half-dollar and a dollar, using 3 pieces at a time?

3. Find the number of triangles which can be formed from 20 points on a plane, no three of which are in the same straight line.

4. A newsboy has 12 daily papers which he sells 3 for 5 cents. In how many ways is it possible to buy 3 different papers?

5. In a society, the offices of president, secretary and treasurer are to be filled. There are ten candidates for the offices; find the number of ways in which a member can make up his ticket.

6. I have the privilege of taking from a collection of 12 books any four I choose. How many selections can I make?

7. How many words, each consisting of 3 vowels and 3 consonants, can be formed from 5 vowels and 4 consonants?

8. How many different throws can be made with three dice?

9. Ten rooms in a recitation building are to be assigned. The department of mathematics requires three rooms. In how many ways can the assignment be made?

10. Find the number of possible assignments of all the rooms, if in the above problem, in addition to the mathematical department requiring three rooms, the department of English requires three, Latin one, German two, and History one.

11. How many changes can be rung with a chime of 10 bells taken all together?

12. 12 boys and 10 girls wish to play at a game requiring 13 players. In how many ways may the players be chosen, if there are to be 6 girls and 7 boys in the game?

13. Find the number of ways in which a tennis set of two ladies and two gentlemen, can be made up from 7 ladies and 8 gentlemen, the gentlemen in the set choosing ladies as partners.

14. If all possible numbers of 5 figures are formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, how many of these numbers will be greater than 45,000?

15. How many words of six letters each can be formed from 4 vowels and 4 consonants, having the first, third and fifth letters of each word consonants, and the other letters vowels?

16. At a game of cards, 6 cards are dealt to each player. Find the number of different hands which any player may have provided there are 52 cards in the pack.

17. A base ball manager has 15 men from which to make up a team of nine. Two are pitchers, three are catchers and the remaining 10 can play in any of the other positions. In how many ways may he make up his nine?

18. In how many ways may 52 cards be dealt to 4 persons, each person to have 13 cards?

CHAPTER XX.

THE BINOMIAL THEOREM.

299. The Binomial Theorem is a formula by the aid of which a binomial may be raised to any power, without actual multiplication.

300. The process of raising a binomial to any power is called *expanding the power of a binomial*; the result obtained is called the *expansion of the power*.

301. Method of Proving the Formula. If the work of expanding the power by actual multiplication is carefully observed, it will be found that the expansion is governed by certain definite laws. The discovery of these laws and their expression in algebraic symbols, will give the *binomial formula*. This method of deriving the formula is called the *method of induction*. It is applied only to the case, in which the exponent of the power is a positive integer.

302. Laws Governing the Expansion. In obtaining the power by actual multiplication, we deal with the case of multiplying together *equal* factors. We can more clearly discern the laws at work in this case, if we consider it in connection with the more general one, in which the second terms of the binomials are all different. Representing these second terms by x_1, x_2, x_3, and so on; and confining our examination, at first, to examples involving the multiplication of two, three and four factors, respectively, we have,

$$(a+x)\ (a+x_1) = a^2 \left.\begin{array}{l} +x \\ x_1 \end{array}\right| a + xx_1. \qquad (1)$$

$$(a+x)\ (a+x_1)\ (a+x_2) = a^3 \left.\begin{array}{l} +x \\ +x_1 \\ +x_2 \end{array}\right| a^2 \left.\begin{array}{l} +xx_1 \\ +xx_2 \\ +x_1x_2 \end{array}\right| a + xx_1x_2. \qquad (2)$$

$$(a+x)\ (a+x_1)\ (a+x_2)\ (a+x_3) = a^4 \left.\begin{array}{l} +x \\ +x_1 \\ +x_2 \\ +x_3 \end{array}\right| a^3 \left.\begin{array}{l} +x\,x_1 \\ +x\,x_2 \\ +x_1x_2 \\ +x\,x_3 \\ +x_1x_3 \\ +x_2x_3 \end{array}\right| a^2 \left.\begin{array}{l} +x\,x_1x_2 \\ +x\,x_1x_3 \\ +x\,x_2x_3 \\ +x_1x_2x_3 \end{array}\right| a + xx_1x_2x_3. \qquad (3)$$

Now making x_1, x_2 and x_3 each equal to x, and reducing each term to its simplest form, we have the expansion, respectively, of the second, third and fourth powers of $a+x$, as follows:

$$(a+x)^2 = a^2+2ax+x^2.$$
$$(a+x)^3 = a^3+3a^2x+3ax^2+x^3.$$
$$(a+x)^4 = a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

By inspecting the products and the resulting powers, we are able to establish for these special powers the following laws:

1. The Law of Exponents.

(*a*) *The first term of the binomial appears in the first term of the expansion, with an exponent equal to the exponent of the required power of the binomial; this exponent is decreased successively by unity in the terms that follow, until in the last term it is zero.*

(*b*) *The exponent of the second term of the binomial begins with zero in the first term of the expansion, and increases by unity in the terms that follow, until in the last term it is equal to the exponent of the required power.*

Proof.

(a) This is proven by noticing the exponents of a in the successive terms of each expansion.

(b) This is proven in a similar way, by noticing the reductions and the exponents of x in the successive terms of each result.

2. The Law of Coefficients.*

(a) *The coefficient of the first term of each expansion is unity.*

(b) *The coefficient of the second term is the exponent of the required power of the binomial.***

(c) *The coefficient of the third term is the number of combinations which can be formed of a number of things equal to the exponent of the power, when two things are taken in each combination.*

(d) *The coefficient of the fourth term is the number of combinations which can be formed of this same number of things, taken three at a time.*

Proof.

(a) This is proven by an inspection of the first term of each expansion.

(b) We observe in equations (1), (2), and (3), that when x_1, x_2 and x_3 are each made equal to x, the coefficients of the powers of a in the second terms of the resulting expansions, become $2x$, $3x$ and $4x$, respectively; or the coefficients of these *terms* (using coefficient as explained in Note 1) are 2, 3 and 4, respectively, being the same in each case as the exponent of the required power.

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* NOTE 1. By coefficients, as here used, is meant, not the coefficients of the powers of x, but the coefficients of the products of the powers of a and x. These are generally called the coefficients of the terms.

** NOTE 2. Since the number of combinations of any number of things taken one at a time, is the same as the number of things, we can write the coefficient of this term equal to the number of combinations which can be made of the number of things denoted by the exponent of the power, taken one at a time. When the exponent is n the coefficient may be written ${}_nC_1$

(c) We also notice in equations (1), (2), and (3), that when the second terms of the binomials are each made equal to x, x^2 will be repeated in the third terms of the expansions, as follows: In (1) as many times as there are combinations of two things taken two at a time; in (2) as many times as there are combinations of three things taken two at a time; and in (3) as many times as there are combinations of four things taken two at a time.

The coefficient of the third term in each case is evidently the number of combinations which can be formed of the number of things denoted by the exponent of the power, when two things are taken in each combination.

(d) This can be proven in a similar way. (Student give proof.)

303. The laws of exponents and coefficients are now established for the second, third and fourth powers of a binomial. We have now to prove the following proposition:

Proposition. *The law of exponents and the law of coefficients will hold for all positive integral powers of a binomial.*

Proof. We will *assume* that the laws given hold for any positive integral power of a binomial, as n.

We will then have:

$$(a+x)^n = a^n + n \text{ or } {}_nC_1\ a^{n-1}x + {}_nC_2 a^{n-2}x^2 + {}_nC_3 a^{n-3}x^3 \dots\dots\dots x^n. \tag{1}$$

Multiplying each member of (1) by $a+x$, we obtain,

$$(a+x)^{n+1} = a^{n+1} + \begin{array}{l|l|l|l|} n \text{ or } {}_nC_1 & a^n x + {}_nC_2 & a^{n-1}x^2 + {}_nC_3 & a^{n-2}x^3 + {}_nC_4 \\ +1 & +{}_nC_1 & +{}_nC_2 & +{}_nC_3 \end{array}\ a^{n-3}x^4 \dots x^{n+1} \tag{2}$$

In the corollary of Art. 297, we have

$$\begin{aligned} {}_nC_1 + {}_nC_2 &= {}_{n+1}C_2. \\ {}_nC_2 + {}_nC_3 &= {}_{n+1}C_3. \\ {}_nC_3 + {}_nC_4 &= {}_{n+1}C_4. \end{aligned}$$

Substituting in (2), and noticing that the new coefficient of the second term is $n+1$, we have

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + {}_{n+1}C_2 a^{n-1}x^2 + {}_{n+1}C_3 a^{n-2}x^3 + {}_{n+1}C_4 a^{n-3}x^4 \dots\dots x^{n+1} \tag{3}$$

This shows that *if the laws are true when $a+x$ is raised to the nth* power, they will be true when $a+x$ is raised to the $(n+1)^{th}$ power.

Let us now, in equation (1), make $n=3$. The equation will give us the third power of $a+x$. We have found by inspection (Art. 302) that the laws hold true for the third power. We have shown in this article that they will then hold when $a+x$ is raised to the $(n+1)^{th}$ or 4th power. By making n equal, in succession to 4, 5, 6, 7 . . . . . $n-1$, we can readily see that the laws hold for $(a+x)^n$, in which n is any *positive integer*.

Hence the proposition is proven.

304. The Binomial Formula. Equation (1), in the previous article, is now proven true. In Art. 295, corollary, we derived,

$$ {}_nC_2 = \frac{n\,(n-1)}{\underline{|2}}, $$

$$ {}_nC_3 = \frac{n\,(n-1)\,(n-2)}{\underline{|3}}, $$

$$ {}_nC_4 = \frac{n\,(n-1)\,(n-2)\,(n-3)}{\underline{|4}}. $$

Substituting these in equation (1), we have the

BINOMIAL THEOREM.

$$ (a+x)^n = a^n + na^{n-1}x + \frac{n\,(n-1)}{\underline{|2}}a^{n-2}x^2 + \frac{n\,(n-1)\,(n-2)}{\underline{|3}}a^{n-3}x^3 \dots x^n \quad (1)^* $$

The following corollaries to the theorem are important:

(a) Applicable when n has any value.

Corollary 1. Coefficients. *The coefficient of any term in the expansion of* $(a+x)^n$ *is equal to the coefficient of the term next preceding it, multiplied by the fraction, whose numerator is the exponent of* a*, in such preceding term, and whose denominator is the exponent of* x *in that term, increased by unity.*

Proof. The corollary is proven by an inspection of the formula.

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* **Note.** While the proof given applies only to the case in which n is a positive integer, the theorem is true for *all* values of n. The proof of the general case involves the use of algebraic principles whose consideration is not included in the design of this work, and it is therefore omitted. The theorem, however, will be *assumed* as proven for the general case, and the formula dealt with, hereafter, under this supposition.

Corollary 2. The General Term. *The general or rth term, in the expansion of $(a+x)^n$ equals*

$$\frac{n\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\,(n-r+2)}{\underline{r-1}}\,a^{n-r+1}x^{r-1}$$

Proof. An examination of the formula shows that the coefficient of any term consists of a fraction, whose numerator is the continued product of $n\,(n-1)$ $(n-2)$ and so on down to a factor, in which the number subtracted from n equals the number of the term diminished by two; and whose denominator is the continued product of the natural numbers from unity to a number one less than the number of the term. Hence the coefficient of the rth term is

$$\frac{(n)\,(n-1)\,(n-2)\ldots\ldots\,(n-[r-2])}{\underline{|r-1}}.$$

Under the law of exponents, governing in the expansion, the exponent of a in each term equals n diminished by one less than the number of the term, hence the exponent of a in the rth term will be $n-(r-1)$; the exponent of x in any term is one less than the number of the term, hence its exponent in the rth term will be $r-1$. We have then,

The rth term $=$

$$\frac{n\,(n-1)\,(n-2)\ldots\ldots(n-r+2)}{\underline{|r-1}}\,a^{n-r+1}x^{r-1} \qquad (2)$$

(b) Applicable only when n is a positive integer.

Corollary 3. Number of Terms. *In the expansion of $(a+x)^n$, there are $n+1$ terms.*

Proof. Since the exponent of x begins with one in the second term and increases successively by unity until, in the last term, it equals n, the expansion evidently contains n terms, *exclusive of the first term*. Adding this term, we have, as the total number of terms in the expansion, $n+1$.

Corollary 4. Equally Distant Terms. *In the expansion of* $(a+x)^n$, *the coefficients of terms equally distant from the beginning and end of the expansion are equal; while the exponents of* a *and* x *are interchanged in the two terms.*

Proof. It is evident that $(a+x)^n=(x+a)^n$. The coefficients in the expansion of $(x+a)^n$ are, therefore, the same as those of $(a+x)^n$, but arranged, in reverse order.

Writing out the expansion in each case, and comparing terms, the corollary is proven.

305. The examples, which follow, will involve the use of the formula and its corollaries. They are separated, for convenience, into two classes:

I. When n is a positive integer.

II. When n is a negative integer; or, a positive or negative fraction.

Class I. When n is a Positive Integer.

Remarks on Examples.

1. When either of the terms of the binomial is other than a single letter, it is best to inclose each term in a parenthesis before using the formula.

Thus, $(2ab+b^2)^5=[(2ab)+(b^2)]^5$.

These terms should first be written in the expansion with their proper exponents and the results obtained, afterwards reduced to their simplest form. (See illustrative example.)

2. The binomial should always be considered as the *sum* of two terms. A minus sign between the terms should be treated as a part of the second term.

Thus $(a-b)^4$ should be written $[(a)+(-b)]^4$.

In the reduction, the even powers of this second term will, of course, be positive, while its odd powers will be negative.

Illustrative Example.

Expand $(3a^2b-2ab^2)^5$ by the binomial theorem.

$(3a^2b-2ab^2)^5=[(3a^2b)+(-2ab^2)]^5=(3a^2b)^5+5(3a^2b)^4(-2ab^2)+10(3a^2b)^3(-2ab^2)^2+10(3a^2b)^2(-2ab^2)^3+5(3a^2b)(-2ab^2)^4+(-2ab^2)^5.=243a^{10}b^5-810a^9b^6+1080a^8b^7-720a^7b^8+240a^6b^9-32a^5b^{10}$.

EXAMPLES.

1. Expand, by the binomial theorem, the following:

1. $(a+x)^7$.
2. $(a-x)^8$.
3. $(1-3a)^6$.
4. $(x^{-\frac{1}{3}}-3x)^5$.
5. $(x^3-2\sqrt{ax})^4$.
6. $\left(x^{-3}-\frac{2}{x}\right)^3$.
7. $(3a^2x-5ax^2)^5$.
8. $(a^{\frac{1}{2}}b^{-\frac{1}{2}}-3ab)^4$.
9. $\left(\frac{2}{\sqrt{y}}+\frac{3}{\sqrt{x}}\right)^6$.
10. $(3a^{-2}x^{\frac{1}{3}}-5ax^{\frac{1}{2}})^4$.
11. $\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^7$.
12. $(2x\sqrt{a}-5y\sqrt[3]{b})$.

Find the term indicated, in the following:

13. The 6th term of $(1+x)^{11}$.

14. The 8th term of $(a-x)^{10}$.

15. The 4th term of $(3a-2x)^7$.

16. The 11th term of $(2a-x^{\frac{1}{3}})^{15}$.

17. The 12th term of $(3a^{-\frac{1}{3}}x^2-ax^{-\frac{1}{3}})^{14}$.

18. The 9th term of $\left(\sqrt{m}-\frac{1}{\sqrt{mx}}\right)^{17}$.

19. The middle term of $(3\sqrt{x}+x^3)^{10}$.

20. The two middle terms of $(3x^{-4}-x^2)^9$.

21. The 4th and 5th terms from the beginning, and from the end of the expansion of $(a-x)^{12}$.

A trinomial can be changed to the form of a binomial and its power expanded by the formula.

Expand the following:

22. $(a+x-y)^4$. 23. $(x-y-z)^3$.

Class II. When n is a negative integer, or a positive or negative fraction.

Remarks on Examples.

The remarks upon the examples in Class I, will apply also to examples in Class II.

The following remarks have special reference to the examples in Class II.

1. Since n is negative or fractional, no one of the factors, n, $n-1$, $n-2$, etc., will reduce to zero. Hence no coefficient will reduce to zero, and the expansion will never terminate.

2. It is often convenient to represent the successive coefficients in the formula by C. C_1, C_2, etc. Their

relative values can then be expressed by the following simple formulas:

$$C = 1. \tag{1}$$

$$C_1 = n. \tag{2}$$

$$C_2 = C_1 \times \frac{n-1}{2}. \tag{3}$$

$$C_3 = C_2 \times \frac{n-2}{3}. \tag{4}$$

The coefficients in a particular example, can readily be obtained by the use of these formulas. If the results are kept in factored form, it will often occur that the law governing the formation of the coefficients in the special example, can be discerned, and much labor saved, in writing out the remaining coefficients. (See second illustrative example.)

Illustrative Examples.

1. Expand $\dfrac{1}{\sqrt{(a-2x^2)^3}}$ to 4 terms.

$$\frac{1}{\sqrt{(a-2x^2)^3}} = [(a)+(-2x^2)]^{-\frac{3}{2}} = a^{-\frac{3}{2}} +$$

$$\left(-\frac{3}{2}\right) a^{-\frac{5}{2}}(-2x^2) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2} a^{-\frac{7}{2}}(-2x^2)^2$$

$$+ \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{2.3} a^{-\frac{9}{2}}(-2x^2)^3 + \ldots\ldots$$

$$= a^{-\frac{3}{2}} + 3a^{-\frac{5}{2}}x^2 + \frac{15}{2}a^{-\frac{7}{2}}x^4 + \frac{35}{2}a^{-\frac{9}{2}}x^6 + \ldots\ldots$$

2. Expand $(m+n)^{-\frac{1}{2}}$ to 5 terms.

Representing the coefficients by C, C_1, C_2, etc.

$$C = 1,$$

$$C_1 = -\frac{1}{2},$$

$$C_2 = C_1 \times \frac{n-1}{2} = \frac{1.3}{2.4},$$

$$C_3 = C_2 \times \frac{n-2}{3} = -\frac{1.3.5}{2.4.6},$$

$$C_4 = C_3 \times \frac{n-3}{4} = \frac{1.3.5.7}{2.4.6.8}.$$

Substituting these coefficients in the formula we have

$$(m+n)^{-\frac{1}{2}} = m^{-\frac{1}{2}} - \frac{1}{2} m^{-\frac{3}{2}} n + \frac{1.3}{2.4} m^{-\frac{5}{2}} n^2 - \frac{1.3.5}{2.4.6} m^{-\frac{7}{2}} n^3 + \frac{1.3.5.7}{2.4.6.8} m^{-\frac{9}{2}} n^4 - \ldots$$

We notice that the factors in both numerator and denominator increase by a common difference of two. Observing this law we can readily write out any number of terms in the expansion.

EXAMPLES.

Expand each of the following to 5 terms:

1. $(a + x)^{\frac{2}{3}}$.
2. $(a - 2b)^{-4}$.
3. $(1 - x)^{\frac{1}{5}}$.
4. $(8a - x)^{-\frac{1}{3}}$.
5. $\sqrt[6]{(x^{\frac{1}{3}} - y^{\frac{1}{2}})^5}$.
6. $(a^5 - 3a^{-\frac{1}{2}})^{-\frac{5}{3}}$.
7. $(2a - 3x^2)^{-6}$.
8. $(ab^{\frac{1}{2}} - a^{\frac{1}{2}}b)^{\frac{2}{3}}$.
9. $\frac{1}{\sqrt[3]{27x^2 - 4m}}$.
10. $\left(1 - \frac{2x}{y}\right)^{-7}$.

11. $\sqrt{(4m^2-2n^{-1})^3}$. 12. $\left(\frac{1}{\sqrt{m}}-\frac{1}{\sqrt[3]{m}}\right)^{-2}$.

In the following, leave the coefficients in the factored form, as in illustrative example, 2.

13. $\sqrt{x-y}$. 14. $\frac{1}{\sqrt{m-n}}$.

15. $\sqrt[4]{y-ax^2}$. 16. $\sqrt{\frac{1}{(x-2a)^3}}$.

Find the term indicated, in the following:

17. The 7th term of $(a+x)^{\frac{2}{3}}$.

18. The 9th term of $(a-2x)^{-\frac{1}{2}}$.

19. The 6th term of $(x^{\frac{1}{2}}-a)^{\frac{1}{4}}$.

20. The 12th term of $\sqrt{x-y}$.

21. The 10th term of $\frac{1}{\sqrt[3]{8a-2}}$.

22. The 8th term of $\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^{-\frac{3}{4}}$.

CHAPTER XXI.

LOGARITHMS.

306. All positive numbers may be considered as powers of some constant number taken as a base.

Thus, representing any number by y, the constant base by a, and the exponent of the power to which a must be raised to equal y, by x, we have $y = a^x$. The exponent x, as here used, is called the *logarithm of y to the base a*. This can be expressed in logarithmic form thus, $x = log_a y$.

307. We may, then, define a logarithm of a number as follows:

Definition of a Logarithm. The logarithm of a number is the exponent of the power to which the given base must be raised to equal the number.

308. Systems of Logarithms. If in the equation

$$y = a^x,$$

we conceive y to represent, in succession, all positive numbers from 0 to $+\infty$, the corresponding values of x will represent the logarithms of these numbers. These logarithms, arranged in the order of the numbers, constitute *a system of logarithms*. Any positive value, except unity, may be assigned to the base a; we may have, therefore, any number of different systems. The two in common use are:

(a) **The Naperian System**, which is the one most generally used in theoretical discussions. In this, $a = 2.71828+$ and is represented by e.

(**b**) **The Common System**, which is the only one used in making numerical computations. In this $a = 10$.

309. It is evident from the definition of a logarithm that the logarithms of the same number in different systems will differ in value. The following, however, are exceptions to this general rule.

Logarithms of a, 1 and 0 in any system taken.

I. *The logarithm of the base itself is 1.*

II. *The logarithm of 1 is 0.*

III. *The logarithm of zero (when the base is greater than 1) is* $-\infty$.

Proof. Take $y = a^x$, and bear in mind that y is the number, a the base, and x the logarithm.

When $x = 1$, $y = a$. Hence I is proven.
When $x = 0$, $y = 1$. Hence II is proven.
When $x = -\infty$, $y = a^{-\infty}$

$$= \frac{1}{a^{\infty}} = 0.$$ Hence III is proven.

If the student, in his future work, is for the moment undecided, as to the logarithms of these quantities, he should recall the definition of logarithms, and ask himself the following questions:

To what power must a be raised to equal a?

To what power must a be raised to equal 1?

To what power must a be raised to equal 0?

310. The forms (1) $y = a^x$ and (2) $x = \log_a y$.

These forms represent the two different ways of symbolizing the definition of a logarithm; they deal with the same quantities and express exactly the same relations. If the definition of a logarithm is held clearly in mind, an equation, given in the one form, can be readily written in the other. This interchange of equations is important, since it often enables the student to employ in his work a logarithmic equation, or the corresponding algebraic one, as the necessities of the case require. (See examples.)

It is evident also from the definition of a logarithm, that equal quantities must have equal logarithms and *vice versa*.

Thus, $m = n$,
hence, $\log_a m = \log_a n$.
Again, $\log_a c^{2y} = \log_a (b+c)$.
hence, $c^{2y} = b+c$.

EXAMPLES.

Change the following to logarithmic equations:

1. $u = e^y$. Ans. $y = \log_e u$.
2. $x = a^m$.
3. $1000 = 10^3$.
4. $625 = 5^4$.
5. $729 = 3^6$.

Change the following to algebraic equations independent of logarithms:

1. $4 = \log_3 81$.
2. $x = \log_a u$.
3. $x+y = \log_e m$.
4. $x = \log_m (x+y)$.

In the practical applications of logarithms, there are four principles of paramount importance involved. These are embodied in the following propositions:

311. Fundamental Propositions Relating to Logarithms.

Prop. I. *The logarithm of any product equals the sum of the logarithms of its factors.*

Proof. In the multiplication of factors, with a common base, the exponent of the base in the product, equals the sum of its exponents in the factors. These *exponents* are, *by definition*, the *logarithms* of the product and factors respectively; *hence the logarithm of the product equals the sum of the logarithms of the factors.*

Prop. II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

Proof. In division, when dividend and divisor have a common base, the exponent of this base in the quotient, equals its exponent in the dividend minus its exponent in the divisor. These *exponents* are, by definition, the *logarithms* of the quotient, dividend and divisor, respectively; *hence the logarithm of the quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

Prop. III. *The logarithm of any power of a quantity equals the logarithm of the quantity multiplied by the exponent of the power.*

Proof. It has been proven that, in raising a quantity to a power, the exponent of the base in the power, equals its exponent in the quantity, multiplied by the exponent of the power. These *exponents*, in the power and the quantity are the *logarithms* of each respectively; *hence the logarithm of the power equals the logarithm of the quantity multiplied by the exponent of the power.*

Prop. IV. *The logarithm of any root of a quantity equals the logarithm of the quantity divided by the index of the root.*

Proof. It has been proven that in extracting the root of a quantity, the exponent of the base in the root equals its exponent in the quantity divided by the index of the root. These *exponents* are, by definition, the *logarithms*, respectively, of the root and the quantity; *hence the logarithm of the root equals the logarithm of the quantity divided by the index of the root.*

Note. It will be excellent practice for the student, to express the proof of each of these propositions in algebraic symbols.

EXAMPLES.

Expand the following:

1. $\log xyz = \log x + \log y + \log z$.

2. $\log 21 = \log 3 + \log 7$.

3. $\log 3mn$. 4. $\log 30$. 5. $\log \frac{xy}{mn}$.

6. $\log \frac{3}{5}$. 7. $\log x^2y^5$. 8. $\log 9a^6$.

9. $\log 36$. 10. $\log \frac{3x^5}{y^2}$. 11. $\log (x^2 - y^2)$.

12. $\log \sqrt{mn} = \log (mn)^{\frac{1}{2}} = \frac{1}{2}(\log m + \log n)$.

13. $\log \sqrt{\frac{m^5n^7}{z^3}}$. 14. $\log a^2\sqrt[3]{a}$.

15. $\log \sqrt{x^2 - y^2}$. 16. $\log a^2\sqrt[3]{x^2 + 2x - 3}$.

17. $\log \frac{1}{5}\sqrt[4]{x^2 - 3x - 4}$. 18. $\log 25\sqrt{\frac{2a^2y}{z^5}}$.

19. $\log \frac{1}{4}\,\frac{2 \times \sqrt[3]{3}}{5 \times \sqrt{4}}$. 20. $\log \sqrt{\frac{x^5y^3}{z}} \times \sqrt[4]{\frac{xy^5}{z^4}}$.

Unite each of the following into one term:

1. $\log a + \log x + \log y = \log axy$.
2. $\log 2 + \log 5$.
3. $\log 2 - \log 5$.
4. $\log (2 + b) - \log m$.
5. $3 \log a + 2 \log c$.
6. $5 \log x - 7 \log y$.
7. $\log 2 + \log 3 + 3 (\log 4 + \log m)$.
8. $\frac{1}{2}\log 2 + \frac{1}{3}\log 3$.
9. $\log 2 + \frac{1}{3}(\log 5 + 4 \log m)$.
10. $\log 7 - \frac{1}{2}\log 3 - \frac{2}{3}\log y$.
11. $\frac{1}{3}(\log 3 + \log 5) - \frac{2}{5}(\log m - \log n)$.
12. $\frac{2}{3}\log m + \frac{1}{3}\log n - \frac{5}{3}\log z$.
13. $\frac{1}{4}\log (x + 1) + \frac{1}{4}\log (x + 2)$.

14. $\frac{1}{3}[\log(x+1)+\log(x-1)]-\log 7$.

15. $2\log 3+3\log 2-\frac{1}{5}\log 7+3\log m$.

312. Relation of Systems of Logarithms. Let us take any two systems of logarithms, representing their bases by a and b; we will represent the logarithm of any number, as m, in the two systems by x and y, respectively; and the logarithm of any other number, as n, by v and r, respectively. We will then have,

$$(1)\qquad m=a^x. \qquad m=b^y.$$
$$(2)\qquad n=a^v. \qquad n=b^r.$$

From (1), $a=m^{\frac{1}{x}}$. $\qquad b=m^{\frac{1}{y}}$.

Substituting in (2), $n=m^{\frac{v}{x}}$. $\qquad n=m^{\frac{r}{y}}$.

Hence, $\frac{v}{x}=\frac{r}{y}$.

Clearing of fractions, $vy=rx$.

Dividing by vx, $\frac{y}{x}=\frac{r}{v}=M$, a constant.

Hence *the logarithms of numbers in one system are equal to the logarithms of the same numbers in another system multiplied by a constant multiplier.*

Corollary. *The constant multiplier, employed in passing from a given system of logarithms to a required system, is the reciprocal of the logarithm of the base of the required system in the given system.*

Take (1) $m=a^x$. $\qquad m=b^y$.

Then, $b=m^{\frac{1}{y}}$.

Substituting m from (1), $b = a^{\frac{x}{y}}$.

Or, $$\frac{x}{y} = \log_a b.$$

Hence, $$\frac{y}{x} = \frac{1}{\log_a b} = M.$$

313. Modulus of Any System. If in the corollary (Art. 312), we take *the Naperian* as the *given* system, then $M = \dfrac{1}{\log_e b}$. This is called the modulus of the required or new system and is represented by m. Hence *the modulus of any system of logarithms is the reciprocal of the logarithm of the base of that system, in the Naperian system.*

In order then to obtain the logarithms of any system, we have simply to multiply the logarithms of the Naperian system by the modulus of the new system.

314. Form of Writing Logarithms. In any system, numbers that are exact powers of the base, will have integral logarithms. The logarithms of intermediate numbers will evidently consist of an integer and a decimal.

Thus, in a system with the base 5, log $5 = 1$, log $25 = 2$, and the logarithm of any number between 5 and 25 will equal 1 plus a decimal. When the logarithm is negative, it is, for convenience, written in the form of a negative integer with a positive decimal. Thus, suppose a logarithm correct to four decimal places is $-(5.2379)$. This is written in the equivalent form $\overline{6}.7621$, the minus sign being placed over the integer, since it alone is negative.

315. Characteristic and Mantissa. The integral part of a logarithm is called its *characteristic* and the decimal part its *mantissa.*

316. Effect of Moving the Decimal Point in the Number. Representing the number by a, and the characteristic and mantissa of its logarithm by c and m respectively, we have, in the *common system*,

$$\begin{aligned}
\log a &= c+m.\\
\log 10a &= \log a+\log 10 = (c+1)+m.\\
\log 100a &= \log a+\log 100 = (c+2)+m.\\
\log 1000a &= \log a+\log 1000 = (c+3)+m.\\
\log \frac{a}{10} &= \log a-\log 10 = (c-1)+m.\\
\log \frac{a}{100} &= \log a-\log 100 = (c-2)+m.
\end{aligned}$$

From which we see that if a number is multiplied or divided by 10 or any multiple of 10, the logarithm of the number and the logarithms of the successive products or quotients will have the same mantissa. Moving the decimal point in a number effects, then, the characteristic of the logarithm but not the mantissa.

317. Rule for Determining the Characteristic in the Common System. As the base of the common system is 10, we have in this system:

$$\begin{aligned}
\log 1 &= 0. & \log 1000 &= 3.\\
\log 10 &= 1. & \log 10000 &= 4.\\
\log 100 &= 2.
\end{aligned}$$

Logarithm of any number

between 1 and 10 equals $0+$a decimal,
between 10 and 100 equals $1+$a decimal,
between 100 and 1000 equals $2+$a decimal,
between 1000 and 10000 equals $3+$a decimal.

Hence the characteristic of any number with one figure to the left of the decimal point is 0; with two figures to the left of the decimal point is 1; with three figures to the left of the decimal point is 2, and so on.

Again,

$$\log 1 = 0. \quad \log .01 = \bar{2}.$$
$$\log .1 = \bar{1}. \quad \log .001 = \bar{3}.$$

Logarithm of any number

between 1 and .1 equals $\bar{1}$+a decimal,
between .1 and .01 equals $\bar{2}$+a decimal,
between .01 and .001 equals $\bar{3}$+a decimal,

and so on.

Hence, in the case of a decimal, the characteristic of the logarithm is negative and one greater than the number of ciphers immediately to the right of the decimal point.

We have, therefore, the following rule for determining the characteristic of the logarithm of any number:

(a) *The characteristic of the logarithm of any number involving an integer, exceeds zero by one less than the number of figures to the left of the decimal point.*

(b) *The characteristic of the logarithm of a decimal fraction is less than zero by one more than the consecutive ciphers immediately to the right of the decimal point.*

318. To Find the Logarithm of any Number. The characteristic is determined by the rule already given. In connection with finding the mantissa from the table of logarithms, two cases arise:

Case I. *When the number contains four figures.*

***Directions**. Find, in the first column at the left of the page, the first three figures of the number; then follow the horizontal line to the right until you come to the column headed, at the top of the page, by the fourth figure; prefix to the four figures here found, the two leading figures, in the column headed 0, which are on the same horizontal line or are the first above it in the zero column; except, when a star is used, then the first leading figures below, in the zero column, are taken. The six figures thus found constitute the mantissa of the logarithm and in connection with the characteristic, form the logarithm of the number.

Example 1. Find the logarithm of 7705.

In the first left-hand column on page 395 we find 770; opposite it in the column headed 5, we find the four figures 6773; the two leading figures on the same horizontal line in the zero column are 88; the characteristic under the rule is 3; hence the entire logarithm is 3.886773.

Example 2. Find the logarithm of 7416.

In column 6, opposite 741 in first column, we find 0170.

Since these figures are starred, we take the leading figures next below in the zero column 87

The characteristic under the rule is 3.

Hence the entire logarithm is 3.870170.

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* Note.—The two specimen pages of a logarithmic table here used (pages 394 and 395) are taken by permission from "Logarithmic Tables" by Prof. George William Jones of Cornell University. Long-continued use of Prof. Jones' tables in class-room work has shown them to be complete, convenient and accurate.

Case II. *When the number includes more than four figures.*

Directions. Find by Case I. the mantissa of the first four figures at the left. Subtract this mantissa from the mantissa of the next higher number and it will give the increase in the logarithm which an increase of one in the fourth figure of the number would produce. The portion of the given number which follows this fourth figure, is a fractional part of unity in this fourth place. Multiply this fraction into the increase in the logarithm produced by adding one to the fourth figure and we have the increase in the mantissa due to the presence of these additional figures in the given number.* Adding this increase to the mantissa of the first four figures and annexing the result to the proper characteristic, we have the logarithm required.

Example 1. Find the logarithm of 79.947.

Logarithm of 79.94 by Case I. = 1.902764.
Mantissa of 79.95 is .902818.
Mantissa of 79.94 is .902764.
Increase in logarithm
due to adding 1 to
fourth figure is 54.
Difference due to 7 in fifth place is
$\frac{7}{10}$ of 54, which approximately = 38.
Total logarithm = 1.902802.

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* NOTE.—It is here assumed that differences in the mantissa will be proportional to differences in the corresponding number. This is true, when the differences are small and will not, in general, cause any error in the first six figures of the number.

If the number following the fourth figure in the above example had been 71, the increase in the mantissa would have been 54 × .71; if the number had been 719, then the increase in the mantissa would have been 54 × .719; and so on.

In order to save the labor of determining these increases or corrections in each example, a column of differences is generally given in connection with logarithmic tables. The differences given are those caused by the presence of the fifth figure in the number. Those corresponding to the sixth, seventh and successive figures are $\frac{1}{10}$, $\frac{1}{100}$....of the tabulated differences.

The use of these tables of differences is illustrated in the following example:

Example 2. Find the logarithm of 723.998.

Log of 723.9 = 2.859679

The difference between this and the next greater logarithm is 60.

Looking in the column of differences, headed 60, we find the difference due to the fifth figure, 9 = 54

Difference due to sixth figure, 8, is $\frac{1}{10}$ of 48 or approximately = 5

Total correction = 59

Required logarithm = 2.859738.

319. Given the Logarithm to Find the Number.

Two cases arise:

Case I. *When the mantissa of the given logarithm is contained in the table.*

Directions. In this case, we have simply to find the given mantissa in the table; prefix to the number at the head of the column, the three figures in the number column at the left which correspond to the mantissa; and then place the decimal point as required by the law of characteristics. This will give us the required number.

Example 1. Find the number corresponding to the logarithm $\bar{1}.887674$.

Number at the head of the column containing the mantissa (page 395) is	1
Number in number column, corresponding to mantisa is	772
Locating the decimal point, the required number is	.7721

Case II. *When the mantissa of the given logarithm is not contained in the table.*

Directions. Find in the table, the mantissa which is next smaller than that of the given logarithm and determine the number (containing four figures) corresponding to this mantissa. Subtract this mantissa from the one next greater in the table; and also from the mantissa of the given logarithm. Now, if the first difference were added to the mantissa of the number found, it would increase the fourth figure of that number by unity; hence the second difference added to the mantissa will increase the fourth figure by such fractional part of unity as the *second* difference is a fractional part of the *first*. We may then find the number corresponding to the given logarithm by adding this fraction to the figure in the fourth place of the number found, or what is equivalent, by writing immediately after such fourth place, the quotient arising from dividing the numerator (the second difference) by the denominator (the first difference); and afterward placing the decimal point as required by the law of characteristics.

Example 1. Find the number corresponding to the logarithm $\bar{3}.856572$.

The next smaller mantissa in the table $= .856548$

The number corresponding to this mantissa $= 7187$

The next greater mantissa in the table $= .856608$

Subtract, $.856548$

Difference corresponding to increase of unity in fourth figure of number found $= 60$

Mantissa of given logarithm $= .856572$

Subtract, $.856548$

Difference in mantissa whose effect on number is to be determined $= 24$

Since a difference of 60 in the mantissa would add unity to fourth figure of the number, a difference of 24 will add $\frac{24}{60}$ to the figure in such fourth place. The equivalent of this, written in the places immediately following the fourth $= 4$

Under the law of characteristics there must be two ciphers between the decimal point and first significant figure; hence number $= .0071874$

In determining the figures which follow the first four found, the table of differences is quite generally employed. The differences given in the table, correspond to figures in the fifth place, but any difference not sufficient to give a significant figure in this place may be multiplied by 10 and the table then used to find the figure in the sixth place, and so on. This use of the table of differences is illustrated in the following example:

Example 2. Find the number corresponding to the logarithm 3.890301.

Next greater mantissa = .890309

Next smaller mantissa = .89 253: corresponding number = 7767

Difference 56

Given mantissa = .890301

Next smaller mantissa = .890253

48

Next smaller difference in column of differences headed 56 = 45; corresponding number = 8

Remainder of difference = 3

Difference for finding sixth figure = 3×10 = 30

Nearest difference in column of differences = 28; corresponding number = 5

Under law of characteristics, there must be four figures to the left of the decimal point; hence number = 7767.85

The work of finding the number corresponding to any logarithm, may be put in condensed form thus,

Example 3. Find the number corresponding to the logarithm 2.901790.

Mantissas,

{ Greater	.901840;	given	.901790		
{ Less	.901785;	less	.901785		7976
Differences,	55		5		
Figures in fifth and sixth places,					09
Number				=	797.609

LOGARITHMS OF NUMBERS.

700	0	1	2	3	4	5	6	7	8	9	Differences.
700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62 61 1 6 6 2 12 12 3 19 18 4 25 24 5 31 31 6 37 37 7 43 43 8 50 49 9 56 55
01	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	
02	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
03	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
04	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
05	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
06	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	
07	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
08	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
09	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	85 1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	60 1 6 2 12 3 18 4 24 5 30 6 36 7 42 8 48 9 54
11	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	
12	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	
13	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
14	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
15	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
16	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
17	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
18	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	
19	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	
720	85 7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	59 1 6 2 12 3 18 4 24 5 30 6 35 7 41 8 47 9 53
21	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
22	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
23	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	
24	9739	9799	9859	9918	9978	*0038	*0098	*0158	*0218	*0278	
25	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
26	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
27	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
28	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
29	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	86 3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	58 1 6 2 12 3 17 4 23 5 29 6 35 7 41 8 46 9 52
31	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
32	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
33	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
34	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
35	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	
36	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	
37	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
38	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
39	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	86 9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	57 1 6 2 11 3 17 4 23 5 29 6 34 7 40 8 46 9 51
41	9818	9877	9935	9994	*0053	*0111	*0170	*0228	*0287	*0345	
42	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	
43	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	
44	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	
45	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
46	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
47	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
48	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	
49	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	
750	0	1	2	3	4	5	6	7	8	9	Differences.

LOGARITHMS OF NUMBERS.

750	0	1	2	3	4	5	6	7	8	9	Differences.
750	87 5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
51	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
52	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	1 6
53	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	2 12
54	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	3 17
											4 23
55	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	5 29
56	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	6 35
57	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	7 41
58	9669	97 6	9784	9841	9898	9956	*0013	*0070	*0127	*0185	8 46
59	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	9 52
760	88 0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
61	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
62	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	1 6
63	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	2 11
64	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	3 17
											4 23
65	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	5 29
66	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	6 34
67	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	7 40
68	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	8 46
69	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	9 51
770	88 6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	
71	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
72	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	1 6
73	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	2 11
74	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	3 17
											4 22
75	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	5 28
76	9862	9918	9974	*0030	*0086	*0141	*0197	*0253	*0309	*0365	6 34
77	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	7 39
78	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	8 45
79	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	9 50
780	89 2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
81	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	55
82	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	1 6
83	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	2 11
84	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	3 17
											4 22
85	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	5 28
86	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	6 33
87	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	7 39
88	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	8 44
89	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	9 50
790	89 7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	
91	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	54
92	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	1 5
93	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	2 11
94	9821	9875	9930	9985	*0039	*0094	*0149	*0203	*0258	*0312	3 16
											4 22
95	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	5 27
96	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	6 32
97	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	7 38
98	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	8 43
99	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	9 49
800	0	1	2	3	4	5	6	7	8	9	Differences.

320. Logarithms may be employed, in accordance with the propositions proven in article 311, in making numerical computations involving multiplication, division, involution and evolution.

Remarks on Examples.

1. In subtracting a greater mantissa from a less, separate the characteristic connected with the less mantissa into two terms, making the second term plus one.

Thus, $4.252647 = 3 + 1.252647$

$2.412354 = 2 \qquad .412354$

Subtracting, we have, 1.840293

Again, $\bar{1}.415639 = \bar{2} + 1.415639$

$3.562115 = 3 \qquad .562115$

Subtracting, we have, $\bar{5}.853524$

After a little experience, the student will be able, even in examples such as the last one given above, to separate the characteristic into the required terms mentally. The process is similar to "borrowing one" in ordinary subtraction.

2. In dividing a logarithm, with a negative characteristic, separate such characteristic into two terms, making the first term negative and the least multiple of the divisor numerically greater than the characteristic; the second term will be a positive number.

Thus, $\bar{3}.152418 \div 6$

$= \bar{6} + 3.152418 \div 6$

$= \bar{1}.525403.$

3. A negative quantity cannot have a logarithm, hence in the application of logarithms to examples in-

volving negative quantities, the sign is ignored, as far as the logarithmic work is concerned, and the sign of the result afterwards determined, as an independent matter, in the same way as if logarithms were not used.

EXAMPLES.

$$\text{Given}\left\{\begin{array}{l}\log 2. = .301030\\ \log 3. = .477121\\ \log 5. = .698970\\ \log 7. = .845098\end{array}\right.$$

Find, from the foregoing, the logarithms of the following:

1. $21.$
2. $1.05.$
3. $.042.$
4. $\frac{2}{3}.$
5. $\frac{6}{7}.$
6. $.3 \div .07.$
7. $1600.$
8. $.0225.$
9. $.63.$
10. $\sqrt{.2}.$
11. $\sqrt[3]{.021}.$
12. $\sqrt[5]{(.14)^3}.$
13. $8\sqrt[3]{.015}.$
14. $\dfrac{2\sqrt{.003}}{9\sqrt[3]{350}}.$

The following examples may be solved by the use of the tables found on pages 394 and 395.

Illustrative Example.

Find by logarithms the value of

$$\frac{(7.247)(-.74967)(.0790054)}{(-70.01)(.77685)}.$$

$$\log\frac{(7.247)(.74967)(.0790054)}{(70.01)(.77685)} = \log\ 7.247 + \log$$

$$.74967 + \log\ .0790054 - (\log\ 70.01 + \log\ .77685).$$

$$\begin{aligned}
\log 7.247 &= 0.860158,\\
\log .74967 &= \bar{1}.874871,\\
\log .0790054 &= \underline{\bar{2}.897657},\\
\log \text{numerator} &= \bar{1}.632686,\\
\log 70.01 &= 1.845160,\\
\log .77685 &= \underline{\bar{1}.890337},\\
\log \text{denominator} &= 1.735497,\\
\log \text{quotient} &= \bar{3}.897189,\\
\text{Quotient} &= .00789204.
\end{aligned}$$

Since the numerator and denominator are each minus, the quotient is positive.

Find by logarithms the value of the following:

1. $\dfrac{(.0794725)(710.56)}{.0077777}$.   2. $(.7)^{20}$.

3. $\sqrt[10]{.0789876}$.

Find the value of the following by the use of any six-place logarithmic table:

1. $(125.67)(.09876)$.

2. $(-.062159)(.00234567)$.

3. $\dfrac{27.63}{214.57}$.   4. $\dfrac{-.017652}{.819536}$.

5. $\dfrac{(.025976)(-.047293)}{(21.869)(-8391.76)(.00479)}$.

6. $(2.7965)^5$.

7. $(-.387621)^3$.

8. $(.678912)^3(-41.625)^2$.

9. $\dfrac{(-6.4217)^5(.0912763)^4}{(-82.7625)^3}$.

10. $\sqrt{3}$.

11. $\sqrt[3]{.25}$.

12. $(.57)^{\frac{3}{5}}$.

13. $(-84.7623)^{\frac{6}{7}}$.

14. $\sqrt[3]{\dfrac{-.012396}{8.12826}}$.

15. $\dfrac{2^{\frac{2}{3}}}{.5^{\frac{4}{7}}}$.

16. $\left(\dfrac{.864207}{.035791}\right)^{\frac{9}{4}}$.

17. $\sqrt[5]{(.129175)(-.087615)}$.

18. $\sqrt[3]{\dfrac{5}{9}}\sqrt[5]{\dfrac{2}{7}}$.

19. $(-.1)^5(-2)^6\sqrt[7]{-3}$.

20. $\dfrac{\sqrt[3]{.127963}}{\sqrt{.0098765}}$.

21. $\dfrac{\sqrt[7]{.00001234}}{(.00198364)^{\frac{3}{5}}}$.

22. $\dfrac{\sqrt[3]{19.67}\sqrt[4]{.5678}}{\sqrt[5]{.0092651}}$.

ANSWERS.

Art. 60. Pages 21 and 22.

1. $4a - 2b - 12 - 3c - d + 4x^2 - 18m$. **2.** $15b + m + 9$.

3. $d - c + k$. **4.** $14\sqrt{x}$.

5. $(a + b + c)\,mx^3 + (b - a - 1)\,nx^2 + (c + a + 3b)x$.

6. $7ab + 7\,(a + b)$. **7.** $9x\sqrt{a} - 6\sqrt[4]{a - b} - 9\,(x - y)$.

8. $(8 - a)\,x^{-2}y^{-1}z^3 + 11m + 4n^am^b + 5a^5 - 2x^c$.

9. $(a + m)\,(x + y) + nz + cd$. **10.** $(c - a)\,(d + b) + (x - m)y$.

11: $(3a + c)\,x^2 + (2 - 6b)\sqrt{a} + (5x - 3m)\,(y + n)$.

12. $(3m + 5n - 2p)y^3 + (2y - 7n)\,(x^2 + \sqrt{a}) + (5a - 4)\,bc + r^2$.

Art. 60. Pages 22 and 23.

1. $2ax - 2by$. **2.** $4ax^2 - (6b - 5a)y^2$.

3. $a^2y - (5a - 3d)\,b^2 + (a + b - c - d)\,m$.

4. $(2d - 2c)\sqrt{7} - 8\,(x + y)\sqrt{2} + 7\sqrt{x}$.

5. $x\sqrt{a - b} + 3m\sqrt{x - y}$.

6. $(2x + 3y)\,m^2 - (4 - 5b)\,n^2 + (p - q)\,z + m$.

Art. 61. Page 24.

1. $3a - b - 3c + 17d$. **2.** $17xy - 4y^2 - 5\sqrt{2} + 4$.

3. $3\sqrt{m - n} + (x^2 - y^2) + 5\sqrt{a - b}$.

4. $6z\sqrt{y - z} - 7xyz + 8abc$. **5.** $(m^{-2} - m^2)\,c + (e + z)\,f + y$.

Art. 61. Page 25.

1. $4\,(a - b) - 2n\,(a + b) - x - y$.

2. $xmn - (6a + b)\,(x + y)^{-8} - mx^2$.

3. $-2x^2 - 4ay + \frac{c}{d}$. **4.** $(4x - 3y)(a + b) - 3bx - \frac{c - d}{m}$.

5. $4a(x^5 - y^3) + (3m + 2a)n(a - h) - az + \frac{c - (b + x)}{z}$.

6. $(a + c - d)(b^2 + y)$. **7.** $2m(x + y)$. **8.** $4ax$.

9. $-6y(5m + 3a)$.

Art. 67. Pages 31 and 32.

1. $2x + a - b + c$. **2.** $m - 5n - 7a + 3$. **3.** $6ab - c - d$.

4. $4a^2 + 3b^2$. **5.** $3m + 2a + 3$. **6.** $6x + 16 + \frac{2x - 1}{9}$.

7. x. **8.** $-5x^3$. **9.** $\frac{(2a + 5)c}{7} - (5x - a)b + 4x$.

10. $(3x - a)(2b - c)(x - y)$. **11.** $4d + 2a$.

12. $a(z + b) - \frac{2y + z}{a + c}$. **13.** 4. **14.** 0. **15.** $5b - 2a$. **16.** 0.

17. a. **18.** $x + y - z + k$.

Art. 81. Pages 43 and 44.

1. $2x^2 + 5xy + 3y^2$. **2.** $a^3 - b^3$. **3.** $9a^2 - 4b^2x^2 + 12bx^3 - 9x^4$.

4. $a^6 - b^6$. **5.** $88m^{-9}n^5 - 37m^{-8}n^4 + 10m^{-7}n^3 + 2m^{-6}n^2$.

6. $x^{3y} - 3x^{2y}z + 3x^yz^2 - z^3$. **7.** $x^{m+n+1} + x^{m+n-2}$.

8. $x^{4p+4} + x^{2p+2}y^{2q} + y^{4q}$.

Art. 82. Pages 46 and 47.

13. $\frac{9}{256}$. **14.** $\frac{1}{270}$. **15.** $12x^6 + 7x^4 + 5x^3 + 10x - 4$.

16. $m^3 + n^3 + r^3 - 3mnr$. **17.** $a^4 - 5a^2b^2 + 4b^4$. **18.** $m^{2n} - x^{2y}$.

19. $x^{-6} + x^{-1}$. **20.** $x^{y+3} + x^{y-2}$. **21.** $a^{2x-1}y^{2c+1} - a^{2x-4}y^{2c+4}$.

22. $cx^2 + c^2x + c^3$. **23.** $a^4 + 4a^2 + 16$. **24.** $-m + m^{\frac{1}{2}} + 20$.

25. $a + 12a^{\frac{2}{3}} + 47a^{\frac{1}{3}} + 60$. **26.** $\frac{9}{25}a^{\frac{6}{5}}b^{-1} - \frac{4}{9}x^{\frac{4}{3}}y^{-\frac{2}{3}}$.

27. $2y^2$. **28.** $8mn^2$. **29.** $9b - 3a$. **30.** -22.

Art. 89. Page 66.

2. $x^2 + 2xy + y^2$. **3.** $a^3 + a^2b + ab^2 + b^3$. **4.** $x^2 - 3x - 2$.
5. $xy + z^c$. **6.** $5a^{x-1} - 2a^{x-2}y + 3a^{x-3}y^2$.

Art. 89. Pages 67 and 68.

10. $\frac{9375}{64}$. **11.** $4a^2 - 5ab + 6b^2$. **12.** $4x^3 + 3x^2 + 2x + 1$.
13. $x^3 - 2x^2 - 3x - 5$. **14.** $x^2 + \frac{3}{4}x - \frac{5}{8}$. **15.** $a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{3}{2}}$.
16. $a^{2x} + 2a^xb^y + b^{2y}$. **17.** $a^{-3}b^2 + 2a^{-2}b^4 - 3a^{-1}b^6$.
18. $x^{a-b}y^c - 2x^ay^{c+d} + 3x^{a+b}y^{c+2d}$.

Art. 99. Pages 84 and 85.

1. $2(x+y)(x+y)$. **2.** $(x-y-3)(x-y-3)$.
3. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.
4. $(x+8)(3x-2)$. **5.** $3ay(x-8)(x+3)$.
6. $(y+2b)(2x-5a)$. **7.** $(2a+1+x)(2a+1-x)$.
8. $(y+3x-5a)(y-3x+5a)$. **9.** $2a(bx+17)(bx-3)$.
10. $(2x-3y+5m-2n)(2x-3y-5m+2n)$.
11. $x^2(x+2)(x^4-2x^3+4x^2-8x+16)$.
12. $(x+6)(x-1)(x-2)(x-3)$.
13. $(x-6)(x+1)(x-4)(x-1)$. **14.** $8(x+2)(2x-1)$.
15. $(3c+2d)(3c-2d)(2a+3b)(2a-3b)$.
16. $(a+3)(a+3)(a-3)(a-3)$.
17. $(3x-2y)(3x-2y-z)$.
18. $(x+y+3)(x+y-3)(x-y+3)(x-y-2)$.
19. $(3a-2b)(3a+b)^4$.
20. $(a^2b^4+4)(ab^2+2)(ab^2-2)(a^2b^4+1)(ab^2+1)(ab^2-1)$.
21. $(x-14)(x-1)(x+2)(x+7)$.
22. $4(2x-3)(2x-3)(x+1)(x+1)$.
23. $(x^a+y^b+z^c)(x^a+y^b-z^c)(x^a-y^b+z)(x^a-y^b-z^c)$.

24. $y^2(x^4+y^4)(x^8-x^4y^4+y^8)$.

25. $(x-v+y-z)(x-v-y+z)$.

26. $(x^4+y^4)(x^2+y^2)(x^2+y^2)(x+y)(x+y)(x-y)(x-y)$.

27. $a^2(a+4)(a^2-4a+16)(a-2)(a^2+2a+4)$.

28. $(a-b)(a-b)(a+2b)$. **29.** $(3y+x)(3y-x+1)$.

30. $(2a-3b)(4a^2+6ab+9b^2-2a-3b)$.

31. $(2a-3)(2a+1)(2a-1)(2a-1)$.

Art. 104. Pages 94 and 95.

3. $9a^3b^2(a-1)$. **4.** x^2-2x+3. **5.** $6ax-8a$. **6.** $a-3$.
7. $2x-3$. **8.** $x-6$. **9.** $6x^2-8x$. **10.** $x+3a$.

Art. 104. Page 95.

1. x^2+2x+3. **2.** $x+1$. **3.** $2x-1$. **4.** $3a-b$. **5.** $2x^2-x$.

Art. 110. Page 100.

2. $a^6-2a^5+4a^4-2a^3+4a^2-2a+3$.

3. $6x^5+7x^4-3x^3+9x^2-9x+2$.

4. $x^7-2x^6+3x^5-x^4+x^3-2x^2+6x-4$.

5. $x^7-5x^6+2x^5+27x^4-87x^3+152x^2-126x+36$.

6. $x^6+11x^5+31x^4-23x^3-160x^2-68x+48$.

Art. 120. Pages 107 and 108.

1. $\frac{3b^2c^2}{5a}$. **2.** $\frac{a^2+ax+x^2}{a+x}$. **3.** $\frac{a+5}{a+3}$. **4.** $\frac{a-1}{a}$.

5. $\frac{c+d}{a-b}$. **6.** $\frac{a+b}{a-2b}$. **7.** $\frac{a+b+c+d}{a-b+c+d}$. **8.** $\frac{x-y+z-m}{x-y}$.

9. $\frac{x-4}{x-5}$. **10.** 1. **11.** $\frac{y+z}{y-z}$. **12.** $\frac{a+b-c}{a}$. **13.** 1.

14. $\frac{x^2+2x-3}{x^2+x-1}$. **15.** $\frac{3x^2-x+2}{x^2-2}$. **16.** $\frac{2x-1}{3x-1}$.

Art. 121. Page 109.

1. x^2+xy+y^2. **2.** $1+\frac{2x}{1-4x}$. **3.** $1-\frac{4x+6}{x^2+2x+3}$.

4. $x - 7 - \frac{3}{x-4}$. **5.** $x^2 + 2xy + y^2$. **6.** $x^a - 4y^b + \frac{y^{2b}}{x^a + 2y^b}$.

Art. 122. Page 111.

1. $\frac{a^2+1}{1+a}$. **2.** $\frac{4x^2 - 1 + 3x}{2x}$. **3.** $\frac{2x^2 - y^2}{x-y}$. **4.** $-\frac{2ab + 2b^2}{a+b} = -2b$.

5. $\frac{1}{x+8}$. **6.** $\frac{x^2 + ax + a^2}{x+a}$. **7.** $\frac{(x+1)^3}{x}$. **8.** $\frac{2}{1+x}$.

Art. 124. Pages 117 to 119.

1. $\frac{a^2c^2 + a^2b^2 - b^2c^2}{abc}$. **2.** $\frac{ay - x^2 - ax}{xy}$. **3.** $\frac{a^2+9}{a^2-9}$.

4. $-\frac{a^2}{a^2-25}$. **5.** $-\frac{3}{(x-2)(x-5)}$. **6.** $\frac{2a^3}{1+a^2+a^4}$. **7.** $\frac{71-59b}{30}$.

8. 3. **9.** $\frac{1+3a^2}{1-a^4}$. **10.** 2. **11.** $-\frac{1}{(x+2)(x-4)}$.

12. $\frac{4}{(a+1)(a+3)}$. **13.** $\frac{4a+24}{(a+1)(a+5)(a+7)}$.

14. $x - 2y + \frac{x^2+y^2}{x^2-y^2}$. **15.** $\frac{2x^3}{1-x^4}$.

16. $\frac{x^2 + 2xy - 4y^2}{(x+y)(x+2y)(x+3y)}$. **17.** 0. **18.** 0. **19.** $\frac{3}{1+9x}$.

20. 0. **21.** $\frac{x+3}{x^4-1}$. **22.** $\frac{x^2+x+1}{x^3(x^2+1)^2}$. **23.** $-\frac{1}{x-3}$.

24. $\frac{2b^2}{x^2-b^2}$. **25.** $ax - 4b^2y - 3by^2 - \frac{2a^2b}{(a-b)^2(a+b)}$.

26. $\frac{1}{(x+7)(x+8)}$. **27.** $\frac{18a-2}{1-9a^2}$. **28.** $\frac{x^2+4xy-y^2}{(x+y)^2(x-y)^2}$.

29. $3y - 2x - \frac{3x^2+xy+2y^2}{x^2-y^2}$. **30.** $\frac{(m-n)^2}{4mn}$.

Art. 125. Pages 122 to 124.

1. $\frac{a^2y^2}{3xz}$. **2.** $\frac{5nx^3}{p^3y}$. **3.** $\frac{x}{2y}$. **4.** $\frac{3(a-b)}{(a+b)^2}$. **5.** $\frac{x-y}{x^2}$.

6. $\frac{9-3x}{z}$. **7.** $\frac{x}{x-1}$. **8.** $-\frac{4+y}{y^2}$. **9.** $\frac{a-7}{a-8}$. **10.** $\frac{c(c^2-a^2)}{a^2}$.

11. 1. **12.** $a^2+1+\frac{1}{a^2}$. **13.** $\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{c^2}{d^2}-\frac{d^2}{c^2}$. **14.** $\frac{a}{a-b}$.

15. 1. **16.** 2. **17.** $\frac{3a^2+3ax+a}{2}$. **18.** $\frac{y^2}{x^2+y^2}$. **19.** $\frac{x-y-z}{xy}$.

20. $\frac{x^2}{y^2}+\frac{2xz}{ay}+\frac{z^2}{a^2}-\frac{y^2}{z^2}$.

Art. 126. Pages 125 to 127.

1. $\frac{4ay^4}{3b^2x}$. **2.** $\frac{x-3}{3}$. **3.** $\frac{1}{x}$. **4.** $\frac{a+1}{a+5}$. **5.** $\frac{x(x+2)}{x-2}$.

6. $\frac{8ax^2-3yz}{2ax-2y+2z}$. **7.** $\frac{a^2x-a^2y+a^3}{xy-ax+ay}$. **8.** $\frac{a-b}{b}$. **9.** $\frac{a-3b}{a-b}$.

10. x^2+x+1. **11.** $\frac{x-p}{x-n}$. **12.** $x^2-1+\frac{1}{x^2}$. **13.** $\frac{x-y-z}{x-y+z}$.

14. 1. **15.** $\frac{a+b-c}{a-b+c}$. **16.** $xy+1+\frac{1}{xy}+\frac{1}{x^2y^2}$.

Art. 127. Pages 128 to 130.

1. $\frac{x^2+3x-10}{x^2+2x-24}$. **2.** $\frac{y+x}{y-x}$. **3.** $\frac{ab-xy}{ab+xy}$. **4.** $\frac{a^2+1}{a^3+2a}$.

5. $\frac{a^2+9}{6a}$. **6.** $1-x-x^2$. **7.** 1. **8.** 1. **9.** $\frac{1}{x+1}$.

10. $\frac{(a^4-x^4)\ (x+a)}{2a^3x^2}$. **11.** $\frac{x+y}{(x-y)^2}$. **12.** $\frac{(x+y)^2}{2y}$. **13.** $\frac{ax}{x+a}$.

Art. 127. Pages 130 and 131.

2. $\frac{(x-y)^2}{(x+y)^2}$. **3.** $\frac{x+1}{2x+1}$. **4.** $\frac{m+n}{n}$. **5.** $-\frac{(a^2+ab+b^2)^2}{ab\ (a^2+b^2)}$.

Art. 143. Pages 143 to 146.

1. 7. **2.** $bc+c+b$. **3.** 1. **4.** 3. **5.** $\frac{8cd-2d}{25c}$. **6.** 6.

7. $b-a$. **8.** $2a$. **9.** $2n$. **10.** $-\frac{69}{17}$. **11.** $\frac{1}{3}$. **12.** 4.

13. $\frac{2}{3}$. **14.** $\frac{n+3p}{2}$. **15.** 1. **16.** $\frac{b+d}{2}$. **17.** 0. **18.** $m+\frac{5}{2}$.

19. $\frac{20ab+4bc+5ac}{4b-5a}$. **20.** $\frac{4bx-4bc}{20b+5c+5x}$. **21.** $\frac{5ax+5ac}{4x-20a-4c}$.

22. $\frac{4bx - 5ax - 20ab}{4b + 5a}$. **23.** $\frac{8m}{11}$. **24.** $-a$. **25.** 5. **26.** $\frac{5}{6}$.

27. $3a$. **28.** -5.

Art. 151. Pages 156 to 162.

1. 32 and 40.
2. $3.
4. 14 days.
5. Boy, 13; girl, 8.
6. 10 years hence.

3. $\begin{cases} \frac{b + c + 2a}{4}, \text{ 1st,} \\ \frac{b + c - 2a}{4}, \text{ 2d,} \\ \frac{c - b}{2}, \text{ 3d.} \end{cases}$

7. 1st. 16, 2d. 7, 3d. 1. **8.** 40. **9.** 1st. 48, 2d. 60.

10. $3000.

11. 1 dollar, 2 half-dollars; 6 quarters, 32 dimes; 41 nickels.

12. 54 minutes. **13.** House, $1700.; lot, $1500.

14. House to grocery, 6 blocks; House to P. O., 8 blocks; P. O. to place of business, 5 blocks.

15. $\frac{ab}{2b - a}$. **16.** 4, 6, 8, 10. **17.** $54\frac{6}{11}$ minutes after 10.

18. $2880. loaned; $1920. invested in losing enterprise. **19.** 468.

20. 8 miles.

21. 1st, $\frac{3b - 22}{19}$. 2d, $\frac{3b + 35}{19}$. 3d, $\frac{b - 1}{19}$. 4th, $\frac{12b - 12}{19}$.

22. $\frac{am - n}{m + 1}$. **23.** 47 gallons. **24.** $11\frac{5}{74}a$. **25.** $\frac{100mc}{ad + ce - ae}$.

26. $15\frac{195}{1427}$ seconds past 6. **27.** A, 1728; B, 2160.

28. $5 - m + \frac{4m}{n}$ hours past noon. **29.** $c + \frac{3m}{80}$. **30.** 160 miles.

31. 1st, 10 miles from A; 2d, 2 miles from A.

32. Current, 2 miles per hour; Boat up stream 3 miles per hour.

Art. 165. Pages 179 to 183.

1. $x = 1, y = 2$. **2.** $x = 1, y = 0$. **3.** $x = \frac{4}{3}, y = \frac{1}{2}$.

4. $x = -3, y = 7$. **5.** $x = \frac{(m + n)^2}{m}, y = -\frac{2mn + n^2}{m}$.

6. $x = 1, y = -1.$ **7.** $x = -1, y = 0.$

8. $x = \frac{ac(2-b)}{1-b}, y = \frac{bc}{1-b}.$ **9.** $x = \frac{2ab}{2ac+b^2}, y = -1.$

10. $x = 2, y = 6.$ **11.** $x = \frac{1}{4}, y = \frac{1}{4}.$ **12.** $x = 3, y = 1.$

13. $x = 1, y = \frac{a+3b}{a-b}.$

14. $x = \frac{1}{mn+pn-mp}, y = \frac{1}{mn+pn-mp}.$

15. $x = 1, y = 2, z = 3.$ **16.** $x = 2, y = 4, z = 6, u = -3.$

17. $x = 1, y = 3, z = 3.$ **18.** $x = \frac{1}{3}, y = \frac{1}{4}, z = \frac{1}{5}.$

19. $x = a, y = a_1, z = a_1.$

20. $x = \frac{2m(n-m)}{n-p}, y = \frac{2m(p-m)}{n-p}, z = 2m.$

21. $x = aa_1a_2, y = a_1a_2 - aa_1 - aa_2, z = a - a_1 - a_2.$

22. $x = 5, y = 4, z = 3.$ **23.** $x = 1, y = \frac{1}{2}, z = \frac{1}{3}, u = \frac{1}{4}.$

24. $x = 1, y = \frac{1}{2}, z = \frac{1}{3}, u = \frac{1}{4}.$ **25.** $x = 3, y = 2, z = 1, u = 4.$

26. $a = \frac{5x}{3}, b = \frac{-50x}{81}, c = \frac{10x}{27}, d = \frac{-34x}{81}.$

27. $x = 1, y = 2, z = 3, u = 4.$ **28.** $x = -abc, y = abc, z = abc.$

Art. 166. Pages 184 to 189.

1. Horse, \$80. Cow, \$30. **2.** \$900., \$600.

3. A, $2b - a + c$; B, $3(a - b - c)$; C, $2c - a + b$.

4. 1st, 300; 2d, 400; 3d, 500. **5.** 742.

6. Horse, \$150. Carriage, \$100. Harness, \$50.

7. Man, 10; boy, 12. **8.** 16 barrels, 20 boxes.

9. A & B, $\frac{1}{3}d(d+1)$ miles; C, $\frac{1}{3}(d+1)(d+2)$ miles.

10. 15 first-class passengers, 20 second-class passengers.

\$2.10, fare for first-class passengers; \$2. fare second-class.

11. John, 15 days; Henry, 10 days.

12. 1st loan, \$1500.; 2d loan, \$500.; Rate, 6%.

13. 6 pounds of 1st, 4 pounds of second mixture.

14. 1st, 20 min.; 2d, 40 min.; 3d, 30 min.

15. 1st kind $\frac{m\,(2n-7)}{n-m}$, second kind $\frac{n\,(2m-7)}{m-n}$.

16. Won 6, lost 4. **17.** Dog $13\frac{1}{2}$ miles, rabbit 12 miles.

18. Man in still water 10 mi. per hour, current 2 miles per hour.

19. Freight 24 miles per hour, passenger 40 miles per hour.

Art. 187. Pages 206 to 208.

1. $x < -2$. **2.** 3 and 4. **3.** $x > 3,\ y > \frac{9}{2}$. **5.** $b > x > c$.

6. $b < x < c$. **7.** $x < 24$. **20.** 25 miles.

21. 6 horses and 12 sheep.

Art. 194. Page 213.

1. $9a^4 b^{-8}$. **3.** $\frac{1}{9} a^{-2} b^{-4} x^2 y^4$.

Art. 206. Pages 219 and 220.

1. $\pm\, 8a^3 b^2$. **3.** $5ab^2$. **9.** $\pm \frac{5a^2c^2x^3}{8m^2 n^4}$. **13.** $\pm\, 8a^{-2}bx^3$.

Art. 208. Page 225.

1. $2x^2 - 3x + 4$. **2.** $a^2 - 3ax + 2x^2$. **3.** $3xy + 4x - 5y^2$.

4. $3bx + x^2 - 2m^2$. **5.** $a^3 - 2a^2b + 3ab^2 - 4b^3$. **6.** $a + b + \frac{1}{c}$.

7. $x^m + y^{\frac{n}{2}} + z^{\frac{P}{3}}$. **8.** $4x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 3$. **9.** $2a^2 - a + \frac{1}{3}$.

10. $2a^{-1} + 3b - 5ab^2$.

Art. 209. Page 230.

1. 428. **2.** 20107. **3.** 6.32. **4.** 2.84604 +. **5.** 198 76.

6. .03507 +. **7.** 1.73205 +. **8.** $\frac{67}{301}$. **9.** .7071 +.

10. .2338 +. **11.** 2.0916 +.

Art. 210. Pages 233 and 234.

1. $a^2 - 2ab + b^2$. **2.** $b^2 + b - 1$. **3.** $3x^2 - 2x - 1$.

4. $x^3 + 2x^2 - 5x + 1$. **5.** $1 - x + x^2 - x^3$. **6.** $m^2 + 3my - 9y^2$.

7. $2x^2y + 5cz - x$.   8. $2a^{\frac{1}{2}} - a^{-\frac{1}{2}} + 3a^{-\frac{3}{2}}$.

9. $(x + a)^{2p}y - 2dbm$.   10. $1 - \frac{x}{3} - \frac{x^2}{9} +$.

Art. 211. Page 238.

1. 427.   2. 1023.   3. $\frac{26}{43}$.   4. 3.41.   5. .4308 +.

6. 1.259 +.   7. .8735 +.   8. 1.481 +.

Art. 212. Page 240.

1. $2a - 3x$.   2. 10.7.   3. $x^2 - 2x - 2$.   4. 517.

5. $3x - 1$.   6. .46.

Art. 229. Pages 252 and 253.

2. $7b\sqrt{6a}$.   3. $28b\sqrt{5ab}$.   4. $a\sqrt{a-b}$.   5. $(x - 3)\sqrt{a}$.

6. $2a^nx^{2n}\sqrt{2}$.   7. $(x + y)\sqrt{x-y}$.   8. $2ab^2\sqrt[3]{7a^2}$.

9. $-27ab\sqrt[3]{2ac^2}$.   10. $9x^my^{2n}\sqrt[3]{10y^2}$.   11. $7a^{3mn}b^{2n}\sqrt[3]{5}$.

12. $8c\sqrt[4]{3}$.   13. $6a^2c\sqrt[5]{a^2c^3}$.   14. $x^2y^{-1}z^m\sqrt[5]{z}$.   15. $(a^2 - b^2)\sqrt{2b}$.

16. $ab\sqrt[n]{a^m}$.   17. $c^2\sqrt[n]{c^n - a^{mn}}$.

Art. 229. Pages 253 and 254.

1. $\frac{3}{5}\sqrt{10}$.   2. $\frac{1}{8}\sqrt{30}$.   3. $\frac{5}{3}\sqrt{3}$.   4. $\sqrt{ab}$.   5. $\sqrt{a^2 - b^2}$.

6. $\frac{1}{b}\sqrt{abc}$.   7. $\sqrt{ac}$.   8. $\frac{c}{2(a - x)}\sqrt{a^2 - ax}$.   9. $\frac{8}{5n}\sqrt[3]{m^2n}$.

10. $\frac{z}{(x + a)x}\sqrt[3]{x^2z^2(x^2 - a^2)}$.   11. $\frac{a}{b^2(a + b)}\sqrt{ab}$.

12. $\frac{3xz}{2}\sqrt[4]{2xz^3}$.   13. $5ab\sqrt[n]{ab^4n}$.

Art. 230. Pages 254 to 256.

1. $(a + b)x^{\frac{1}{2}}$.   2. $(ab + c + d)x^{\frac{1}{3}}$.   3. $(3 + m)(a - b)^{\frac{1}{5}}$.

4. $(cy + d)y^{\frac{1}{2}}$.   5. $2m^2(m - n)^{\frac{1}{3}}$.   6. $\frac{2x^3 + 5a^2x}{8}(x^2 + a^2)^{\frac{1}{3}}$.

7. $\frac{1+5x^6}{45}(1-x^3)^{\frac{1}{2}}$. **8.** $12\sqrt{3}$. **9.** $57\sqrt{2}$. **10.** $(a+c)\sqrt[m]{b}$.

11. $5\sqrt[3]{3}$. **12.** $(3a+5)ab\sqrt{2ab}$. **13.** $9\sqrt[3]{2a}$. **14.** $2\frac{2}{7}\sqrt{21}$.

15. $9cm\sqrt{y-z}$. **16.** $(3a-1)\sqrt{5m}$. **17.** $-3\sqrt{x^2-a^2}$.

18. $(m+y^2zx)\sqrt[x]{mc}$. **19.** $(3x^{-1}y^2+2x^2y+x^{a+3})\sqrt[3]{2x^a}$.

20. $(4b^pc^n+3b^{3-n}c+2x^2)\sqrt[n]{b^3c^5}$. **21.** $\frac{b^2-2a+ab}{x}\sqrt{5a^mb}$.

Art. 231. Pages 256 and 257.

1. $(a-b)x^{\frac{1}{3}}$. **2.** $(b-cx)(m-n)^{\frac{1}{5}}$. **3.** $(c-2d)(a^2-3x)^{\frac{1}{3}}$.

4. $2b(a-b)^{\frac{1}{2}}$. **5.** $4\sqrt{7}$. **6.** $7\sqrt{2}$. **7.** $2\sqrt[3]{5}$.

8. $(2a-5b)\sqrt{7x}$. **9.** $-\frac{7}{6}\sqrt[3]{11}$. **10.** $\frac{7xy}{12b}\sqrt[3]{4xyb^2}$.

11. $-\frac{267}{26}\sqrt{5}$. **12.** $\frac{2ab}{x^2-1}\sqrt[n]{x^2-1}$.

Art. 232. Pages 258 and 259.

1. $15a^{\frac{5}{6}}$. **2.** $12xy$. **3.** $(4a^2)^{\frac{2}{3}}$ or $(2a)^{\frac{4}{3}}$. **4.** $(6xy)^{\frac{3}{2}}$.

5. $x^{\frac{7}{6}}y^{\frac{5}{4}}z^{\frac{17}{12}}$. **6.** $(x-y)^{\frac{2}{3}}(x+y)^{\frac{3}{4}}$ or $(x^2-y^2)^{\frac{2}{3}}(x+y)^{\frac{1}{12}}$.

7. $24\sqrt{3}$. **8.** $\sqrt{a^2-x^2}$. **9.** $\sqrt[6]{72}$. **10.** $12a^2bx^2\sqrt[6]{2a}$.

11. $5a^2b\sqrt[6]{3a}$. **12.** $24a^2c^2\sqrt[12]{81a^9b^7c}$. **13.** $\frac{ax}{by}\sqrt[4]{b^2cxy^2}$.

14. $10b^6cx^2y\sqrt[6]{32xy^9}$. **15.** $\frac{a^4-x^4}{2x^2}\sqrt[12]{432x^5}$. **16.** $6x^3y^2\sqrt[mn]{a^nx^my^m}$.

17. $\frac{1}{20}\sqrt{5}$. **18.** $(2ab)^x$. **19.** $\sqrt[6n]{x^{19}}$. **20.** $\frac{a^2c}{y^2}\sqrt[6]{\frac{ab^3}{x^3y}}$.

21. $\frac{a}{z^2}\sqrt[4]{ab^2y^2z^2}$.

Art. 232. Pages 259 and 260.

1. $x-y$. **2.** $a^{\frac{5}{4}}-a-5a^{\frac{3}{4}}+7a^{\frac{1}{2}}-2a^{\frac{1}{4}}$. **3.** a^4-1. **4.** 1.

5. $5-2\sqrt{6}$. **6.** $57+12\sqrt{15}$. **7.** 2. **8.** 5. **9.** $a-c^2\sqrt[3]{b^2}$.

10. x^4+1. **11.** $5\sqrt{2}+2+2\sqrt[4]{8}+5\sqrt[3]{2}+\sqrt[6]{32}+2\sqrt[12]{128}$.

Art. 233. Pages 261 and 262.

1. $a^{\frac{1}{6}}b^{\frac{2}{15}}$. **2.** $2(ab)^{\frac{1}{6}}$. **3.** $5x^{\frac{7}{15}}y^{-\frac{1}{15}}$. **4.** $2^{-\frac{1}{2}}$. **5.** $(4a-3x)^{\frac{1}{2}}$.

6. $b(a^2x)^{\frac{1}{n}}$. **7.** $a^{\frac{3n-m}{mn}}c^{\frac{n-3m}{mn}}$. **8.** $12\sqrt{2}$. **9.** $15a^3x^2\sqrt[15]{4a^7}$.

10. $2x^{2-m}m\sqrt[6]{\frac{16}{3}}$. **11.** $6a^3x^2\sqrt[15]{64a^{-1}b^7}$. **12.** $7a\sqrt[10]{a^3bc^{-1}}$.

13. $\sqrt[4]{z^{-1}-x^{-1}}$. **14.** $\sqrt[6]{\frac{4m^2y^3}{anz^3}}$. **15.** $\sqrt[12]{\frac{x^4}{a^3}}$.

Art. 233. Pages 262 and 263.

1. $a^{\frac{2}{5}}+4a^{\frac{1}{5}}+5$. **2.** $a^{\frac{1}{3}}-4a^{\frac{1}{6}}b^{\frac{1}{6}}+16b^{\frac{1}{3}}$. **3.** $a^{\frac{2}{3}}-2a^{\frac{1}{3}}b^{\frac{1}{3}}$.

4. $\frac{3}{2}\sqrt{10}-4$. **5.** $4\sqrt[3]{2}-8\sqrt[6]{2}+2\sqrt[12]{2}$.

6. $2\sqrt{c}+3\sqrt{x}$. **7.** $\sqrt{2}-\sqrt[4]{2}$. **8.** $2\pm\sqrt{3}$.

9. $\sqrt{x}-\sqrt[4]{xy}+\sqrt{y}$. **10.** 1. **11.** $\frac{1}{x}$.

Art. 234. Page 264.

1. $\frac{16}{81}a^{\frac{4}{3}}$ **2.** $(a+b)^2$. **3.** $9(x-y)(a+b)^{\frac{3}{2}}$.

4. $9\sqrt[3]{9}$. **5.** $\frac{a^6}{216}\sqrt[3]{6a^2}$. **6.** $16a^6x^4\sqrt[3]{9a^2b^2}$. **7.** $\frac{a^4}{243}\sqrt{3a}$.

8. $\frac{b^6x^6}{a^6y^6}\sqrt[4]{\frac{bx}{ay}}$. **9.** $9a^2b^2c\sqrt{2a}$. **10.** $(x-z)^2y^3\sqrt[3]{y(x-z)}$.

11. $11-6\sqrt{2}$. **12.** $11\sqrt{2}-9\sqrt{3}$. **13.** $1+\sqrt{1-x^2}$.

Art. 235. Page 266.

1. $3(3)^{\frac{1}{6}}$. **2.** $\frac{2}{3}a^{\frac{1}{6}}$. **3.** $5^{\frac{1}{m}}(a+b)^{\frac{1}{2m}}$. **4.** $x(6-2x)^{\frac{1}{3}}$.

5. $\left(\frac{xy^2}{az^2}\right)^{\frac{1}{7}}$. **6.** $\frac{1}{2}\sqrt[6]{2}$. **7.** $4ax^3\sqrt[4]{3x^2}$. **8.** $n\sqrt[4]{3m^2n^3}$.

9. $\frac{1}{2}\sqrt{2x}$. **10.** $3x\sqrt[3]{2}$. **11.** $\frac{1}{5}\sqrt[3]{20}$. **12.** $3a^2\sqrt[8]{2x}$.

13. $a\sqrt[10]{b}$.

Art. 242. Pages 269 and 270.

1. $4ab+\sqrt{-a}$. **2.** $5a-8\sqrt{-b}$. **3.** $-20\sqrt{-1}$.
4. $17\sqrt{-1}$. **5.** $(18a^2+2b+6c^3)\sqrt{-1}$.
6. $(4a+9ab)\sqrt{-2ab}$. **7.** $6ax\sqrt{-(a+b)}$.

Art. 243. Page 270.

1. $3\sqrt{-1}$. **2.** $6\sqrt{-1}$. **3.** $4a\sqrt{-1}$. **4.** $a\sqrt{b}+b\sqrt{-1}$.
5. $216a^3xy\sqrt{-1}$. **6.** $(a^2-x^2)\,3b\sqrt{-1}$.

Art. 243. Pages 272 and 273.

1. $-\sqrt{15}$. **2.** -12. **3.** $-ab$. **4.** $6\sqrt{10}$.
5. $2\sqrt{30}$. **6.** 1. **7.** $-10ab\sqrt{3a}$. **8.** $\frac{-6x}{x-y}$.
9. $-\frac{1}{2(a-b)}\sqrt{a-b}$. **10.** a^2+b. **11.** 12.
12. $\sqrt{10}+2\sqrt{-6}-\sqrt{-15}+6$. **13.** $(a+c)\sqrt{-1}$.
14. $-3-2\sqrt{3}-\sqrt{15}$. **15.** $24-\sqrt{6}-3\sqrt{10}+4\sqrt{15}$.
16. $-5-2\sqrt{6}$. **17.** $-48+24\sqrt{3}$. **18.** -1. **19.** 1.
20. 27. **21.** 27. **22.** -8.

Art. 244. Page 274.

1. $3\sqrt{-2}$. **2.** 4. **3.** $\frac{7}{2}\sqrt{2}+2\sqrt{5}$.
4. $\frac{8}{3}\sqrt{-6}+\frac{2}{3}\sqrt{6}-3\sqrt{-1}$.
5. $x\sqrt{-a}+a\sqrt{-c}$. **6.** $am\sqrt{x}-n^2\sqrt{yz}$.

Art. 244. Pages 275 and 276.

2. $-\sqrt{-1}$. **3.** $-\sqrt{-a}$. **4.** $-2\sqrt{-3}$. **5.** $-\frac{2}{3}\sqrt{-3}$.
6. $\sqrt{-5}$. **7.** $-\sqrt{-1}$. **8.** $-\frac{a}{b}\sqrt{-1}$. **9.** $4\sqrt{-1}+\sqrt{5}$.

10. $-\frac{5}{3}\sqrt{-3}-1.$ **11.** $-\sqrt{-3}-\sqrt{-1}-\sqrt{2}.$

12. $-3ab\sqrt{-6}+x^3\sqrt{2}+2a^2b\sqrt{-3}.$

13. $-a\sqrt{y-x}-8b\sqrt{-x-y}.$

Art. 247. Page 278.

2. $\frac{7+2\sqrt{10}}{3}.$ **3.** $\frac{a^2-2a\sqrt{b}+b}{a^2-b}.$ **4.** $\sqrt{a^4-1}-a^2.$

5. $\frac{1+2\sqrt{-6}}{5}.$ **6.** $-\frac{2\sqrt{5}+2\sqrt{3}+\sqrt{30}+3\sqrt{2}}{2}.$

7. $\frac{1+\sqrt{1-x^2}}{x}.$ **8.** $\frac{2-\sqrt{6}+\sqrt{2}}{4}.$ **9.** $\frac{\sqrt{6}+\sqrt{15}}{3}.$

Art. 248. Page 279.

2. $\frac{x^{\frac{5}{3}}-x^{\frac{4}{3}}y^{\frac{1}{6}}+xy^{\frac{1}{3}}-x^{\frac{2}{3}}y^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{2}{3}}-y^{\frac{5}{6}}}{x^2-y}.$

3. $\frac{a^3b^{\frac{2}{3}}+a^{\frac{5}{2}}b^{\frac{4}{3}}+a^2b^2+a^{\frac{3}{2}}b^{\frac{8}{3}}+ab^{\frac{10}{3}}+a^{\frac{1}{2}}b^4}{a^3-b^4}.$

4. $-(17+8\sqrt[6]{72}+8\sqrt[3]{9}+12\sqrt{2}+12\sqrt[3]{3}+6\sqrt[6]{648}).$

5. $\frac{a^{\frac{9}{2}}-a^4b^{\frac{1}{5}}+a^{\frac{7}{2}}b^{\frac{2}{5}}-a^3b^{\frac{3}{5}}+a^{\frac{5}{2}}b^{\frac{4}{5}}-a^2b+a^{\frac{3}{2}}b^{\frac{6}{5}}-ab^{\frac{7}{5}}+a^{\frac{1}{2}}b^{\frac{8}{5}}-b^{\frac{9}{5}}}{a^5-b^2}.$

Art. 249. Page 280.

1. 7.2426 +. **2.** 1.3866 +. **3.** .3592 +. **4.** 5.4845 +.

Art. 252. Page 284.

2. $5-2\sqrt{2}.$ **3.** $7+3\sqrt{2}.$ **4.** $1+\frac{1}{2}\sqrt{2}.$

5. $2\sqrt{7}-\sqrt{14}.$ **6.** $(a-b)-2\sqrt{ab}.$ **7.** $2-\sqrt{-3}.$

8. $1-\sqrt{-1}.$ **9.** $\frac{c}{2}+\frac{\sqrt{a^2-c^2}}{2}.$

10. $\frac{\sqrt{3abd^2-a^2b^2c^2}}{bd}+\frac{ac}{d}.$ **11.** $\frac{1}{2}\sqrt{4b^2-8ab+a^2}+\sqrt{ab}.$

Art. 254. Pages 286 and 287.

1. 4. **2.** 5. **3.** 7. **4.** -3. **5.** $\frac{n^2}{4m}-m$. **6.** 49. **7.** $\frac{1}{2}$. **8.** 3. **9.** 2.

10. $16m$. **11.** -20. **12.** $\frac{3}{4}\left[\frac{(c-1)^2}{c}\right]$ **13.** a^2. **14.** 16. **15.** 2.

16. 4. **17.** 1.

Art. 255. Pages 288 to 290.

1. $(m^{\frac{3}{4}}-n^{\frac{3}{4}})\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}}$. **2.** 1024. **3.** 1. **4.** $x^{-\frac{3}{2}}y^{\frac{79}{24}}$.

5. $2y\sqrt{y^2-1}$. **6.** $\frac{e^y}{\sqrt{e^{2y}-4}}$. **7.** $\frac{1}{x}\sqrt{mx+x^2}$.

8. $\frac{1}{2x(1-x^2)+\sqrt{1-x^2}}$. **9.** $-\frac{2}{x^2(x^3+1)^{\frac{5}{3}}}$.

10. $\frac{2}{\sqrt{(x^2+a^2)(x^2+b^2)}}$. **11.** $-\frac{1}{8x\sqrt{1-x}}$. **12.** 0.

Art. 259. Pages 292 to 294.

1. ± 4. **2.** $\pm 2\sqrt{5}$. **3.** $\pm\frac{1}{2}$. **4.** ± 4. **5.** $\pm\frac{4}{3a}$. **6.** ± 5.

7. $\pm\frac{1}{3}b\sqrt{3}$. **8.** 0. **9.** $\pm\frac{4}{5}$. **10.** $\pm\frac{1}{2}\sqrt{b}$. **11.** ± 2.

12. $\pm\frac{1}{2}\sqrt{-14}$. **13.** ± 2. **14.** ± 3. **15.** $\pm m$.

16. $\pm\frac{3a+b}{a^2+4b^2}\sqrt{a^2+4b^2}$. **17.** $\pm\frac{a}{2(1+c)}\sqrt{1+c}$. **18.** ± 10.

Art. 261. Page 299.

1. $-\frac{1}{4}$ or $-\frac{5}{4}$. **2.** 1 or $-\frac{7}{3}$. **3.** $\frac{1\pm\sqrt{29}}{14}$. **4.** $\frac{7}{6}$ or $-\frac{1}{2}$.

5. $\frac{3}{4}$ or $-\frac{1}{4}$. **6.** $\frac{5}{9}$ or $-\frac{1}{3}$. **7.** $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

8. $\frac{c}{m}$ or $-\frac{9c}{5m}$. **9.** $\frac{1}{4d}$ or $\frac{1}{12d}$. **10.** $\frac{1}{3a}$ or $\frac{1}{18a}$.

11. $2-\sqrt{2}$ or $1-\sqrt{2}$. **12.** $2+2\sqrt{5}$ or $1-\sqrt{5}$.

Art. 262. Page 301.

1. $\frac{1}{2}$ or $-\frac{2}{3}$. **2.** $\frac{7}{3}$ or $\frac{5}{4}$. **3.** $\frac{1 \pm \sqrt{-34}}{5}$. **4.** $\frac{3}{2}$ or 0.

5. 1 or $-\frac{21}{19}$. **6.** $\frac{2}{a}$ or $-\frac{5}{7a}$. **7.** $\frac{1}{a+b}$ or $\frac{1}{b-a}$.

8. $\frac{2a}{3b}$ or $-\frac{3b}{2a}$. **9.** $\frac{c}{5}$ or $\frac{d}{b}$. **10.** $\frac{m}{n}$ or $-\frac{5}{c}$.

Art. 265. Pages 306 and 307.

1. Real and equal. **2.** Imaginary.

3. Real and unequal. **4.** Real and unequal.

5. Real and unequal when $m > 12$ numerically. Real and equal when $m = \pm 12$. Imaginary when $m < 12$ numerically.

6. Real and unequal when $c < 3$ numerically. Real and equal when $c = 3$. Imaginary when $c > 3$ numerically.

7. Real and unequal for all real values of g.

8. $b > \sqrt{15m}$ numerically, and m positive. **11.** $r = \pm\frac{3}{5}\sqrt{5}$.

12. Real; positive; $x_1 > x_2$. **13.** Imaginary.

14. Real; negative; equal. **15.** Real; x_1 positive, $x_2 = 0$; $x_1 > x_2$.

16. Real; x_1 positive, x_2 negative; $x_1 > x_2$.

17. Real; x_1 positive, x_2 negative; $x_2 > x_1$.

18. Real; $x_1 = 0$, x_2 negative; $x_2 > x_1$. **19.** Real; positive; equal.

20. Real; negative; $x_2 > x_1$.

Art. 266. Pages 307 to 309.

1. 9 or 3. **2.** $\frac{3}{2}$ or -1. **3.** $\frac{13 \pm \sqrt{33}}{4}$. **4.** $\frac{-3 \pm \sqrt{5}}{2}$.

5. -2 or -6. **6.** 7 or 3. **7.** $\frac{-1 - 2n \pm \sqrt{4m^2 + 4n + 1}}{2}$.

8. 4 or -2. **9.** $-\frac{1}{3}$ or -3. **10.** 12 or 4. **11.** 1 or $-\frac{39}{5}$.

12. 4 or -1. **13.** 7 or $-\frac{1}{3}$. **14.** $5\frac{1}{3}$ or 3. **15.** $\frac{19}{8}$ or $\frac{5}{4}$.

16. 10 or -6. **17.** $\frac{b \pm \sqrt{b^2 + 4ab - 4bc}}{2}$.

18. $\frac{9 - 5\sqrt{3}}{2}$ or $\frac{7\sqrt{3} - 13}{2}$. **19.** b or $-\frac{5}{3}b$.

20. $\frac{1-m}{n}$ or $\frac{n}{m-1}$. **21.** $1 + \sqrt{2}$ or $-2 - 2\sqrt{2}$.

22. a or $-\sqrt{b}$. **23.** $ab - a + 1$ or a. **24.** $6m$ or a.

25. $\frac{m}{2b}$ or $\frac{m - 4a}{2b}$. **26.** 17 or 2. **27.** $\frac{2a + 3b}{a + b}$ or $\frac{3b - 2a}{a - b}$.

28. 4 or -6. **29.** $\frac{a+b}{a-b}$ or $\frac{b-a}{a-3b}$. **30.** $16a^4 - m$ or $a^4 - m$.

Art. 267. Pages 309 to 313.

1. 5, 4 or -4, -5. **2.** 10, 5 or -5, -10. **3.** 20 by 80 rods.

4. $\frac{2}{3}$. **5.** 24. **6.** current, 2; crew, 10 miles per hour.

7. 8 inches square. **8.** $1.50. **9.** 12. **10.** 60. **11.** 16.

12. 15. **13.** 30 or 16 hours. **14.** 10 ft. **15.** $30.

16. A, 8 and B, 10 miles per hour. **17.** 10. **18.** 8 miles.

19. 40 and 62½ acres. **20.** 5 cts. and 4½ cts. per lb.

21. 48 or 8 miles. **22.** 4%. **23.** 120 gal. **24.** 520 mi.

Art. 269. Pages 319 to 321.

1. ± 3 or $\pm \sqrt{5}$. **2.** $\sqrt[5]{2}$ or 1. **3.** $\frac{1}{2}$ or $-\frac{5}{4}$. **4.** 1 or $\sqrt[m]{-\frac{5}{8}}$.

5. $\pm \frac{1}{2}$ or $\pm \frac{1}{3}\sqrt{-6}$. **6.** 729 or $\frac{64}{729}$. **7.** 5, -1, or $2 \pm \sqrt{2}$.

8. 1, -2, 3, or -4. **9.** ± 4. **10.** 1, 2, or $-3 \pm \sqrt{7}$. **11.** 2 or 4.

12. 3, -5, or $-1 \pm \sqrt{12 + \sqrt[3]{49}}$. **13.** $\frac{8}{3}$ or $-\frac{1}{3}$.

14. $\frac{1 \pm \sqrt{105}}{4}$, 2 or $-\frac{3}{2}$. **15.** $\pm\sqrt{2}$ or ± 2. **16.** 2 or $\frac{81}{8}$.

17. 8, -2, or $3 \pm 3\sqrt{10}$. **18.** -1, -2, or -3. **19.** $-\sqrt[5]{\frac{3}{2}}$ or -1.

20. 4 or $\frac{367}{125}$.

21. $1, -\frac{1}{3}$ or $\frac{1 \pm 2\sqrt{7}}{3}$

22. 9^n, or $\left(\frac{121}{25}\right)^n$.

23. $2, -\frac{7}{3}$ or $\frac{-2 \pm \sqrt{562}}{12}$.

24. $3, -\frac{5}{8}$ or $-3 \pm \sqrt{-31}$.

25. 1 or $\frac{1}{16}$.

26. $\pm\frac{1}{2}, -2$ or -1.

27. $1, \frac{9}{4}$ or $-\frac{1}{4}$.

28. $\frac{1}{4}\sqrt[3]{2}$, or $-\frac{2}{121}\sqrt[3]{44}$.

29. $1, -2$, or $\frac{-1 \pm \sqrt{-15}}{2}$.

30. 3 or $3 \pm 2\sqrt{2}$.

31. $\pm\frac{1}{16}\sqrt{2}$ or $\pm 27\sqrt{-3}$.

32. $3, 2$, or $\frac{5 \pm \sqrt{-3 + 8\sqrt{2}}}{2}$.

33. 0 or $\frac{5 \pm \sqrt{-39}}{8}$.

34. $\left(\frac{1}{3}\right)^{\frac{4a}{3}}$ or $\left(-\frac{5}{9}\right)^{\frac{4a}{3}}$.

35. $1, -4$, or $\frac{-3 \pm \sqrt{281 + 192\sqrt{2}}}{2}$.

36. $2, -\frac{7}{2}$ or $\frac{-9 \pm \sqrt{1409}}{12}$.

37. $2m^4$ or $m^4(1+5\sqrt[3]{5})$.

38. $\pm\sqrt{10}$ or $\pm\sqrt{-10}$.

39. $1, -\frac{1}{4}$ or $\frac{-3 \pm \sqrt{-19}}{14}$.

40. $6, 1$ or $\frac{7}{2} \pm \frac{1}{14}\sqrt{1285}$.

41. $m + n^2$ or $m + 81n^2$.

Art. 272. Pages 326 to 328.

1. 0 or $-\frac{1}{3}$.

2. 2 or 3.

3. $3, -2$ or $\frac{1 \pm \sqrt{37}}{2}$.

4. $\frac{3}{2}$ or ± 2.

5. 0 or $\pm\frac{2}{3}$.

6. $\pm 3, -5$ or 2.

7. $\frac{2a}{3}$ or $\frac{-a \pm a\sqrt{-3}}{3}$.

8. ± 1 or 2.

9. $\frac{3}{2}$ or $\frac{1}{6}$.

10. $\pm 1, \pm 2, -5$, or 3.

11. $\pm a, \frac{-a \pm a\sqrt{-3}}{2}$ or $\frac{a \pm a\sqrt{-3}}{2}$.

12. $\pm 2, \pm 1$, or $-\frac{3}{2}$.

13. $\pm a$ or $\pm 3a$.

14. $-\frac{6}{7}$ or $\frac{7}{9}$.

15. $\pm 1, 3$ or 5.

16. $\pm\sqrt{2}, \pm\sqrt{-2}$ or 3.

17. $\frac{1}{n}$ or $-\frac{1}{m}$. **18.** $m \pm m\sqrt{-1}$ or $-m \pm m\sqrt{-1}$.

19. $\frac{3}{2}$ or $\frac{4}{5}$. **20.** $-\frac{17}{5}$ or $-\frac{13}{4}$. **21.** $-a$ or $\frac{3}{2}$.

22. $\frac{a}{b}$ or $-\frac{2a}{b}$. **23.** $\pm 2c$ or $\pm\frac{1}{2}c$. **24.** $\frac{3}{4}$ or $-\frac{5}{2}$.

25. $-6, 3$ or 0. **26.** $-\frac{7}{2}$ or $-\frac{5}{2}$. **27.** ± 1 or $\pm\frac{1}{2}$.

28. $\frac{1 \pm \sqrt{-11}}{2}$ or $\frac{-1 \pm \sqrt{-11}}{2}$. **29.** $\frac{3}{2}, \frac{2}{3}$ or $-\frac{6}{13}$.

30. $3, 4$ or $\frac{7}{2}$. **31.** 1, or $-\frac{1}{2} \pm \frac{1}{2a}\sqrt{4a - 3a^2}$. **32.** 0 or ± 3.

33. $1 \pm \sqrt{-1}$ or $-1 \pm \sqrt{-1}$. **34.** $2, -5$ or 9.

Art. 274. Pages 332 and 333.

1. $(2x-4)(2x+3)$. **2.** $(2x-3)(x+8)$. **3.** $(7x-1)(2x-3)$.

4. $(5-x)(x+1)$. **5.** $(3x-2-\sqrt{5})(3x-2+\sqrt{5})$.

6. $(\sqrt{7}-1-x)(\sqrt{7}+1+x)$.

7. $2(x-1-\sqrt{-1})(x-1+\sqrt{-1})$.

8. $(mx-n)(mx-2n)$. **9.** $(1-2mx)(1+3mx)$.

10. $(ax+b)(cx+2b)$. **11.** $(3x-a-b)(5x-a+b)$.

12. $(x+y-5)(x-2y+3)$. **13.** $(x-2z+3)(x+2z+3)$.

14. $(1-y-x)(2+3y+x)$. **15.** $(2x-3a+1)(x-2a-1)$.

16. $(3a-4b+1)(2a-b+2)$. **17.** $(x-y-z)(x+2y+3z)$.

18. $(xy-2)(xz+3)$.

Art. 279. Pages 336 and 337.

1. $x = 1$ or 3. **2.** $x = 5$ or -2.
$y = 3$ or 1. $y = 2$ or -5.

3. $x = 2$ or $-\frac{1}{5}$. **4.** $x = \frac{69}{7}$ or 5. **5.** $x = \frac{25}{8}$ or 4.
$y = 1$ or $\frac{49}{5}$. $y = -\frac{47}{7}$ or 3. $y = \frac{25}{6}$ or 3.

6. $x = -4$ or 4. **7.** $x = 8$ or -1.
$y = 6$ or 2. $y = 4$ or 1.

Art. 281. Page 338.

1. $x = 0, \quad 0, \quad 5 \text{ or } -\frac{44}{9}.$
$y = 20, -22, -5 \text{ or } \frac{44}{9}.$

2. $x = \pm 1, 2 \text{ or } -\frac{1}{3}.$
$y = 0, 2 \text{ or } -\frac{1}{3}.$

3. $x = -4, \frac{5}{4}, \quad 2 \text{ or } -\frac{5}{2}.$
$y = 1, -\frac{5}{16}, \quad 1 \text{ or } -\frac{5}{4}.$

4. $x = -\frac{22}{17}, 1, 1 \text{ or } -\frac{11}{13}.$
$y = -\frac{88}{17}, 4, -5 \text{ or } \frac{55}{13}.$

Art. 282. Page 341.

1. $x = \pm 2 \text{ or } \pm \frac{1}{3}\sqrt{3}.$
$y = \pm 1 \text{ or } \mp \frac{5}{3}\sqrt{3}.$

2. $x = \pm 2 \text{ or } \pm \frac{5}{13}\sqrt{-26}.$
$y = \pm 4 \text{ or } \mp \frac{7}{13}\sqrt{-26}.$

3. $x = \pm 3 \text{ or } \pm \frac{1}{3}\sqrt{3}.$
$y = \pm 2 \text{ or } \mp \frac{8}{3}\sqrt{3}.$

4. $x = \pm 1 \text{ or } 0.$
$y = \pm 2 \text{ or } \pm \sqrt{3}.$

5. $x = 1, -3, \text{ or } \pm \frac{13}{19}\sqrt{19} - 1.$
$y = 0, -2, \text{ or } \mp \frac{9}{19}\sqrt{19} - 1.$

6. $x = \pm 2, \pm 2\sqrt{-1}, \text{ or } \pm \sqrt[4]{\frac{20}{11}}.$
$y = +1, -1, \qquad \text{ or } -\sqrt[6]{\frac{245}{11}}.$

Art. 283. Page 344.

1. $x = 3 \text{ or } 4.$
$y = 4 \text{ or } 3.$

2. $x = \pm 4 \text{ or } \pm 6.$
$y = \pm 6 \text{ or } \pm 4.$

3. $x = 1 \text{ or } 3.$
$y = 3 \text{ or } 1.$

4. $x = 4 \text{ or } 1.$
$y = 1 \text{ or } 4.$

5. $x = \pm 2 \text{ or } \pm \sqrt{-1}.$
$y = 1 \text{ or } \sqrt[3]{-4}.$

6. $x = \pm 2.$
$y = \pm 1.$

Art. 284. Page 346.

1. $x = 1, 2 \text{ or } \frac{-5 \pm \sqrt{17}}{2}$.
$y = 2, 1 \text{ or } \frac{-5 \mp \sqrt{17}}{2}$.

2. $x = \pm 1$.
$y = \pm 2$.

3. $x = 14, 2, \text{ or } \frac{55 \pm 3\sqrt{-311}}{8}$.
$y = 5, 1 \text{ or } \frac{-3 \pm \sqrt{-311}}{8}$.

4. $x = \pm 1 \text{ or } \pm \sqrt{31}$.
$y = 2 \text{ or } -38$.

Art. 285. Pages 347 to 349.

1. $x = 7 \text{ or } -4$.
$y = 2 \text{ or } -\frac{7}{2}$.

2. $x = \pm 3$.
$y = \pm 2$.

3. $x = 6 \text{ or } -5$.
$y = 1 \text{ or } -\frac{6}{5}$.

4. $x = 5 \text{ or } -\frac{14}{5}$.
$y = 1 \text{ or } -\frac{97}{20}$.

5. $x = 1 \text{ or } 2$.
$y = 2 \text{ or } 1$.

6. $x = 2 \text{ or } 2 \pm \sqrt{-10}$.
$y = 2 \text{ or } 2 \mp \sqrt{-10}$.

7. $x = \pm 3 \text{ or } \pm \frac{7}{6}$.
$y = \pm 4 \text{ or } \mp \frac{3}{2}$.

8. $x = 1, 5 \text{ or } 3 \pm \sqrt{-58}$.
$y = 5, 1 \text{ or } 3 \mp \sqrt{-58}$.

9. $x = 3 \text{ or } -2$.
$y = 2 \text{ or } -3$.

10. $x = 1 \text{ or } -2$.
$y = 2 \text{ or } -1$.

11. $x = \pm 6$.
$y = \pm 1$.

12. $x = 2, 4 \text{ or } -1 \pm \frac{1}{3}\sqrt{33}$.
$y = 4, 2 \text{ or } -1 \mp \frac{1}{3}\sqrt{33}$.

13. $x = 3, 6 \text{ or } -2 \pm \sqrt{-14}$.
$y = 2, 1, \text{ or } \frac{-2 \mp \sqrt{-14}}{3}$

14. $x = \pm 2$.
$y = \pm 1$.

15. $x = 1 \pm \sqrt{-1} \text{ or } 0$.
$y = 1 \mp \sqrt{-1} \text{ or } 0$.

16. $x = 5, -3, \text{ or } -2 \pm \sqrt{13}$.
$y = 3, -5, \text{ or } 2 \pm \sqrt{13}$.

17. $x = 2, 6, \text{ or } 0$.
$y = 6, 2, \text{ or } 0$.

18. $x=3,\ 2$ or $\frac{5\pm\sqrt{-51}}{2}$.

$y=2,\ 3$ or $\frac{5\mp\sqrt{-51}}{2}$.

19. $x=a-b$ or $b-a$.

$y=a+b$ or $-a-b$.

20. $x=3,-\frac{7}{3},\ -3$ or $\frac{11}{3}$.

$y=2,-2-\frac{8}{5}$ or $\frac{8}{5}$.

21. $x=4$ or $\frac{100}{21}$.

$y=9$ or $\frac{169}{21}$.

22. $x=\frac{1}{3},1$ or $\frac{1\pm\sqrt{-11}}{6}$.

$y=1,\frac{1}{3}$ or $\frac{1\mp\sqrt{-11}}{6}$.

23. $x=\pm 2$ or $\pm\frac{6}{7}$.

$y=\pm 1$ or $\mp\frac{57}{7}$.

24. $x=3$ or $-5\pm 2\sqrt{2}$.

$y=2$ or $2\pm 2\sqrt{2}$.

25. $x=\pm 2$ or ± 1.

$y=\pm 1$ or ± 2.

26. $x=\pm 1$ or $\pm\frac{13}{18}$.

$y=\mp 2$ or $\mp\frac{83}{36}$.

27. $x=\pm 3$ or $\pm\sqrt{2}$.

$y=\pm\sqrt{2}$ or ± 3.

28. $x=1,2,\frac{3+\sqrt{10\pm\sqrt{-15}}}{2},\frac{3-\sqrt{10\pm\sqrt{-15}}}{2}$, or $\frac{3\pm\sqrt{7}}{2}$.

$y=2,1,\frac{3-\sqrt{10\pm\sqrt{-15}}}{2},\frac{3+\sqrt{10\pm\sqrt{-15}}}{2}$, or $\frac{3\mp\sqrt{7}}{2}$.

29. $x=0,\ b-a$, or $\frac{a+3b\pm\sqrt{a^2+2ab+5b^2}}{2}$.

$y=0,\ a-b$, or $\frac{a-b\mp\sqrt{a^2+2ab+5b^2}}{2}$.

30. $x=\pm 3,\pm 2,\ \frac{3\sqrt{2}\pm\sqrt{-6}}{2}$ or $\frac{-3\sqrt{2}\pm\sqrt{-6}}{2}$.

$y=\pm 2,\ \pm 3,\ \frac{3\sqrt{2}\mp\sqrt{-6}}{2}$ or $\frac{-3\sqrt{2}\mp\sqrt{-6}}{2}$.

31. $x = \pm 3$ or $\pm \frac{6}{157}\sqrt{157}$.
$y = \pm 3$ or $\mp \frac{21}{157}\sqrt{157}$.

32. $x = 3$ or $27 \pm \frac{2}{3}\sqrt{-546}$.
$y = -3$ or $-27 \pm \frac{2}{3}\sqrt{-546}$.

33. $x = 4, -3, \frac{9 \pm \sqrt{-31}}{2}, \frac{9 \pm \sqrt{17}}{2}$ or $\frac{1 \pm \sqrt{97}}{2}$.
$y = 3, -4, \frac{-9 \pm \sqrt{-31}}{2}, \frac{-9 \pm \sqrt{17}}{2}$ or $\frac{-1 \pm \sqrt{97}}{2}$.

34. $x = 3, -\frac{11}{3}$ or $\frac{-1 \pm \sqrt{705}}{8}$.
$y = 1, -\frac{11}{9}$ or $\frac{-1 \pm \sqrt{705}}{64}$.

35. $x = \pm 3$.
$y = \pm 1$.

36. $x = 5, \frac{41}{3}, 7$ or $\frac{35}{3}$.
$y = 4, -\frac{40}{3}, 0$ or $-\frac{28}{3}$.

37. $x = 8$ or 1.
$y = 1$ or 8.

38. $x = 3, 1$ or $\frac{3 \pm \sqrt{-7}}{2}$.
$y = 1, 3$ or $\frac{3 \mp \sqrt{-7}}{2}$.

39. $x = 2$ or $-\frac{57}{10}$.
$y = 1$ or $\frac{43}{10}$.

40. $x = 2, -1$ or $2 \pm \sqrt{5}$.
$y = 1, -2$ or $-2 \pm \sqrt{5}$.

41. $x = 2, -\frac{2}{5}$ or $\frac{4 \pm 5\sqrt{2}}{5}$.
$y = 1, \frac{3}{5}$ or $\frac{4 \mp 5\sqrt{2}}{5}$.

Art. 286. Pages 349 to 353.

1. ± 3 and ± 5. 2. ± 5 and ± 3 or $\mp \frac{4}{3}$ and $\pm \frac{37}{6}$. 3. 12 or 21.

4. $\frac{1}{3}$. 5. 4%. 6 yrs. 6. 20 by 80 rods.

7. 800 apples, $\frac{3}{8}$ cts. each.

8. A, 10; and B, $7\frac{1}{2}$ mi. per hour. A. 80, and B. 60 miles. Met in 8 hours.

9. $\frac{x \pm \sqrt{2y - x^2}}{2}, \frac{x \mp \sqrt{2y - x^2}}{2}$.

10. A, 4 or 6 mi. an hr. in 9 or 12 hrs., respectively. B, 6 or 8 mi. an hr. in 6 or 9 hrs., respectively. C, 12 mi. in 3 or 6 hrs. Distance 36 or 72 miles.

11. Fore-wheel, 12 ft.; hind-wheel $13\frac{1}{3}$.

12. 4%, rate. 104%, buying price. **13.** 16 or $16\frac{1}{3}$ square rods.

14. $25 per room. A, 6; and B, 4 rooms.

15. 1st, 15 da. at $2; 2d, 10 da. at $3.

16. 100 mi. 40 and 60 mi. per hr.

17. $\frac{88}{15}$ rev. per sec.; 8 and 10 mi.

18. 15 and 20 yds. per sec. 150 yds.

19. A, 30; and B, 20 lbs.; 600 lbs. wheat. **20.** A, $300; B, $150.

Art. 291. Page 358.

2. 720; 720; 2441880. **3.** 358800. **4.** 720. **5.** 5040.

Art. 296. Pages 360 and 361.

2. 3003; 50388; 15504. **3.** 15. **4.** 20. **5.** 4845.

Art. 298. Pages 363 and 364.

2. 20. **3.** 1140. **4.** 220. **5.** 720. **6.** 495. **7.** 28800.

8. 56. **9.** 120. **10.** 50400. **11.** 3628800. **12.** 166320.

13. 1176. **14.** 16800. **15.** 576. **16.** 20358520.

17. 3628800. **18.** $\dfrac{\underline{|52}}{[\underline{|13}]^4}$.

Art. 305. Pages 373 and 374.

1. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$.

2. $a^8 - 8a^7x + 28a^6x^2 - 56a^5x^3 + 70a^4x^4 - 56a^3x^5 + 28a^2x^6 - 8ax^7 + x^8$.

3. $1 - 18a + 135a^2 - 540a^3 + 1215a^4 - 1458a^5 + 729a^6$.

4. $x^{-\frac{5}{3}} - 15x^{-\frac{1}{3}} + 90x - 270x^{\frac{7}{3}} + 405x^{\frac{11}{3}} - 243x^5$.

5. $x^{12}-8x^{\frac{19}{2}}a^{\frac{1}{2}}+24x^7a-32x^{\frac{9}{2}}a^{\frac{3}{2}}+16a^2x^2$.

6. $x^{-9}-6x^{-7}+12x^{-5}-8x^{-3}$.

7. $243a^{10}x^5-2025a^9x^6+6750a^8x^7-11250a^7x^8+9375a^6x^9-3125a^5x^{10}$.

8. $a^2b^{-2}-12a^{\frac{5}{2}}b^{-\frac{1}{2}}+54a^3b-108a^{\frac{7}{2}}b^{\frac{5}{2}}+81a^4b^4$.

9. $\frac{64}{y^3}+\frac{576}{y^{\frac{5}{2}}x^{\frac{1}{2}}}+\frac{2160}{y^2x}+\frac{4320}{y^{\frac{3}{2}}x^{\frac{3}{2}}}+\frac{4860}{yx^2}+\frac{2916}{y^{\frac{1}{2}}x^{\frac{5}{2}}}+\frac{729}{x^3}$.

10. $81a^{-8}x^{\frac{4}{3}}-540a^{-5}x^{\frac{3}{2}}+1350a^{-2}x^{\frac{5}{3}}-1500ax^{\frac{11}{6}}+625a^4x^2$.

11. $x^{\frac{7}{2}}y^{-\frac{7}{2}}-7x^{\frac{5}{2}}y^{-\frac{5}{2}}+21x^{\frac{3}{2}}y^{-\frac{3}{2}}-35x^{\frac{1}{2}}y^{-\frac{1}{2}}+35x^{-\frac{1}{2}}y^{\frac{1}{2}}-21x^{-\frac{3}{2}}y^{\frac{3}{2}}$
$+7x^{-\frac{5}{2}}y^{\frac{5}{2}}-x^{-\frac{7}{2}}y^{\frac{7}{2}}$.

12. $8x^3a^{\frac{3}{2}}-60ab^{\frac{1}{3}}x^2y+150a^{\frac{1}{2}}b^{\frac{2}{3}}xy^2-125by^3$. **13.** $462x^5$.

14. $-120a^3x^7$. **15.** $-22680a^4x^3$. **16.** $96096a^5x^{\frac{10}{3}}$.

17. $-9828a^{10}x^{\frac{7}{3}}$. **18.** $24310m^{\frac{1}{2}}x^{-4}$. **19.** $61236x^{\frac{35}{2}}$.

20. $30618x^{-12}-10206x^{-6}$.

21. $-220a^9x^3+495a^8x^4$ and $495a^4x^8-220a^3x^9$.

22. $a^4+4a^3x-4a^3y+6a^2x^2-12a^2xy+6a^2y^2+4ax^3-12ax^2y$
$+12axy^2-4ay^3+x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

23. $x^3-3x^2y-3x^2z+3xy^2+6xyz+3xz^2-y^3-3y^2z-3yz^2-z^3$.

Art. 305. Pages 376 and 377.

1. $a^{\frac{2}{3}}+\frac{2}{3}a^{-\frac{1}{3}}x-\frac{1}{9}a^{-\frac{4}{3}}x^2+\frac{4}{81}a^{-\frac{7}{3}}x^3-\frac{7}{243}a^{-\frac{10}{3}}x^4$.

2. $a^{-4}+8a^{-5}b+40a^{-6}b^2+160a^{-7}b^3+560a^{-8}b^4$.

3. $1-\frac{1}{5}x-\frac{2}{25}x^2-\frac{6}{125}x^3-\frac{21}{625}x^4$.

4. $2^{-1}a^{-\frac{1}{3}}+\frac{2^{-4}}{3}a^{-\frac{4}{3}}x+\frac{2}{9}2^{-7}a^{-\frac{7}{3}}x^2+\frac{14}{81}2^{-10}a^{-\frac{10}{3}}x^3+\frac{35}{243}2^{-13}a^{-\frac{13}{3}}x^4$.

5. $x^{\frac{5}{18}}-\frac{5}{6}x^{-\frac{1}{18}}y^{\frac{1}{2}}-\frac{5}{72}x^{-\frac{7}{18}}y-\frac{35}{1296}x^{-\frac{13}{18}}y^{\frac{3}{2}}-\frac{455}{31104}x^{-\frac{19}{18}}y^2$.

6. $a^{-\frac{25}{3}} + 5a^{-\frac{83}{6}} + 20a^{-\frac{58}{3}} + \frac{220}{3}a^{-\frac{142}{6}} + \frac{770}{3}a^{-\frac{91}{3}}.$

7. $\frac{1}{64a^6} + \frac{9x^2}{64a^7} + \frac{189x^4}{256a^8} + \frac{189x^6}{64a^9} + \frac{5103x^8}{512a^{10}}.$

8. $a^{\frac{2}{3}}b^{\frac{1}{3}} - \frac{2}{3}a^{\frac{1}{6}}b^{\frac{5}{6}} - \frac{1}{9}a^{-\frac{1}{3}}b^{\frac{4}{3}} - \frac{4}{81}a^{-\frac{5}{6}}b^{\frac{11}{6}} - \frac{7}{243}a^{-\frac{4}{3}}b^{\frac{7}{3}}.$

9. $\frac{x^{-\frac{2}{3}}}{3} + \frac{4x^{-\frac{8}{3}}m}{3^5} + \frac{32x^{-\frac{14}{3}}m^2}{3^9} + \frac{896x^{-\frac{20}{3}}m^3}{3^{14}} + \frac{8960x^{-\frac{26}{3}}m^4}{3^{18}}.$

10. $1 + \frac{14x}{y} + \frac{112x^2}{y^2} + \frac{672x^3}{y^3} + \frac{3360x^4}{y^4}.$

11. $8m^3 - 6mn^{-1} + \frac{3}{4}m^{-1}n^{-2} + \frac{1}{16}m^{-3}n^{-3} + \frac{3}{256}m^{-5}n^{-4}.$

12. $m + 2m^{\frac{7}{6}} + 3m^{\frac{4}{3}} + 4m^{\frac{3}{2}} + 5m^{\frac{5}{3}}.$

13. $x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}y - \frac{1}{2.4}x^{-\frac{3}{2}}y^2 - \frac{1.3}{2.4.6}x^{-\frac{5}{2}}y^3 - \frac{1.3.5}{2.4.6.8}x^{-\frac{7}{2}}y^4.$

14. $m^{-\frac{1}{2}} + \frac{1}{2}m^{-\frac{3}{2}}n + \frac{1.3}{2.4}m^{-\frac{5}{2}}n^2 + \frac{1.3.5}{2.4.6}m^{-\frac{7}{2}}n^3 + \frac{1.3.5.7}{2.4.6.8}m^{-\frac{9}{2}}n^4.$

15. $y^{\frac{1}{4}} - \frac{1}{4}y^{-\frac{3}{4}}ax^2 - \frac{3}{4.8}y^{-\frac{7}{4}}a^2x^4 - \frac{3.7}{4.8.12}y^{-\frac{11}{4}}a^3x^6$

$- \frac{3.7.11}{4.8.12.16}y^{-\frac{15}{4}}a^4x^8.$

16. $x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}(2a) + \frac{3.5}{2.4}x^{-\frac{7}{2}}(2a)^2 + \frac{3.5.7}{2.4.6}x^{-\frac{9}{2}}(2a)^3$

$+ \frac{3.5.7.9}{2.4.6.8}x^{-\frac{11}{2}}(2a)^4.$

17. $-\frac{91}{6561}a^{-\frac{16}{3}}x^6.$ 18. $\frac{6435}{128}a^{-\frac{17}{2}}x^8.$ 19. $-\frac{231}{8192}x^{-\frac{19}{8}}a^5.$

20. $-\frac{4199}{2^{19}}x^{-\frac{21}{2}}y^{11}.$ 21. $\frac{135850}{3^{13}}2^{-19}a^{-\frac{28}{3}}.$ 22. $\frac{129789}{4^9}x^{-\frac{59}{8}}y^{\frac{59}{8}}.$

Art. 320. Page 397.

1. 1.322219. 2. .021189. 3. $\bar{2}.623249$. 4. $\bar{1}.823909$.

5. $\bar{1}.933053$. 6. .632023. 7. 3.204120. 8. $\bar{2}.352182$.

9. $\bar{1}.799340$. **10.** $\bar{1}.650515$. **11.** $\bar{1}.440739$. **12.** $\bar{1}.487677$.
13. .295120. **14.** $\bar{3}.237326$.

Art. 320. Page 398.

1. 7260.5. **2.** .000797922. **3.** .77581.

Art. 320. Pages 398 and 399.

1. 12.4112. **2.** —.000145804. **3.** .128769. **4.** —.021539.
5. .0000013975. **6.** 171.031. **7.** —.05824. **8.** 542.184.
9. .00000133715. **10.** 1.73205. **11.** .62996. **12.** .713715.
13. 44.9515. **14.** —.115105. **15.** 2.35887. **16.** 1292.4.
17. —.408085. **18.** .639874. **19.** .000748755.
20. 5.0706. **21.** 8.32302. **22.** 5.9765.

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